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Research Article

# Optimal Fourth- and Eighth-Order Iterative Solver and Their Basins of Attraction

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**Abstract.** We developed a new, fourth- and eighth-order optimal approach for solving nonlinear equations in this study. With three function evaluations, the new methods' convergence order is four; with four function evaluations, it is eight. Furthermore, according to the Kung-Traub hypothesis, it is optimal. In comparison to the suggested approaches, numerical results are provided to verify the superior computing efficiency of the current robust methods. We examine a wide range of practical issues, including projectile velocity to verify the suitability and efficacy of our suggested approaches. Lastly, in order to illustrate their dynamic behaviour on the complex plane, the basins of attraction are also provided.

**Keywords.** Basins of attraction, Multi-point iterations, Optimal order, Non-linear equation

**Mathematics Subject Classification (2020).** 41A25, 65D05

## 1. Introduction

The nonlinear equation $f(x) = 0$ is one of the biggest issues facing scientific computers, engineering, and applied mathematics in general. The method that is most frequently used to solve nonlinear equations is Newton's iteration method. Newton's approach has been refined by other academics to achieve higher order convergence and more accurate findings, see e.g., Cordero *et al.* [10], Curry *et al.* [11], Huang *et al.* [12], Madhu [16], Nadeem *et al.* [17], Soleymani *et al.* [23], Vrscay [25], Vrscay and Gilbert [26]. In addition, the *Efficiency Index* (*EI*) is a widely used technique to evaluate the effectiveness of various iterative approaches. The definition of this index is $p^{1/m}$, where $m$ is the number of functional evaluations required at each iteration and $p$ is the convergence order. If and only if the iterative method with $m$ functional evaluations has an order of convergence equal to $2^{m-1}$, according to the conjecture of Kung and Traub [14]. The most effective iterative methods for varied convergence orders have been developed by numerous scholars. Typically, the composition methodology is used to construct an optimal method, along with a few approximations and interpolations to minimise the amount of functional evaluations needed at each iteration. Various optimal fourth order and eighth order iterative techniques were developed, see e.g., Abdullah *et al.* [1,2], Wang and Li [27,28]. Further, we studied the behaviour of iterative scheme in the complex plane. Furthermore, a number of researchers have applied these concepts to many iterative schemes (Amat *et al.* [3,4], Cordero *et al.* [10], Curry *et al.* [11], Soleymani *et al.* [23], Tao and Madhu [24], Vrscay [25], Vrscay and Gilbert [26]), which discussed the basin of attraction of a few well-known iterative schemes.

The rest of the paper is set up as follows. The proposed strategies have been developed and their convergence analysis is covered in Section 2. The performance of the proposed approaches and other comparison methods is shown in Section 3 and is supported by numerical examples. Solve a real-world applications in Section 4 to demonstrate the efficacy of the suggested techniques. Section 5 uses basins of attraction to study the suggested methods in the complex plane. Section 6 provides concluding observations.

## 2. Construction of Proposed Methods

We will define an *Iterative Function* (*IF*) by $x_{n+1} = \psi(x)$. Using the additional information at $x, \phi_1(x), \dots, \phi_i(x)$, $i \ge 1$, let $x_{n+1}$ be calculated. Nothing from the past is utilised. Consequently,

$$x_{n+1} = \psi(x, \phi_1(x), \dots, \phi_i(x)). \tag{2.1}$$

A multipoint *IF* without memory is then defined as $\psi$.

The Newton-Raphson (also known as Newton-*IF*) ($NR_2$) is provided by

$$\psi_{NR_2}(x) = x - u(x), \quad u(x) = \frac{f(x)}{f'(x)}. \tag{2.2}$$

With two function evaluations, the ($NR_2$) *IF* is a one-point *IF* that meets the Kung-Traub conjecture for $d = 2$. Also, $EI_{NR_2} = 1.414$.

### 2.1 Proposed Optimal Fourth Order *IF*

In this way, we attempt to derive a new optimal fourth order *IF*,

$$\left.\begin{aligned} &\psi_{SSDM_4}(x) = \psi_{NR_2}(x) - H(\tau)\frac{f(\psi_{NR_2}(x))}{f'(x)}, \\ &H(\tau) = H(1) + (\tau-1)H'(1) + \frac{1}{2}(\tau-1)^2 H''(1) + \dots \quad \text{and} \quad \tau = 1 - \frac{f(\psi_{NR_2}(x))}{f(x)}. \end{aligned}\right\} \tag{2.3}$$

The next theorem addresses the selection of the parameter $|H''(1)|$ for which the suggested (2.3) approach has the best fourth order convergence.

**Theorem 2.1.** *Assume that the function $f: D \subset \mathbb{R} \to \mathbb{R}$ has continuous derivatives and is suitably smooth. If $x_0$ is selected in a suitably small neighborhood of $x^*$ and $f(x)$ has a simple root $x^*$ in the open interval $D$, then the approach* (2.3) *has fourth order convergence, when*

$$H(1) = 1, \quad H'(1) = -2, \quad |H''(1)| < \infty. \tag{2.4}$$

*The error equation is satisfied,*

$$e_{n+1} = \left(\left(5 - \frac{H''(1)}{2}\right)c_2^3 - c_2c_3\right)e^4 + O(e^5), \tag{2.5}$$

$$c_j = \frac{f^{(j)}(x^*)}{j!f'(x^*)}, \quad j = 2,3,4,\dots \quad \textit{and} \quad e = x - x^*.$$

*Proof.* Let $\tilde{e} = \psi_{NR_2}(x) - x^*$, $\hat{e} = \psi_{SSDM_4}(x) - x^*$. Extending $f(x)$ and $f'(x)$ around $x^*$ using Taylor's technique, we have

$$f(x) = f'(x^*)(e + c_2e^2 + c_3e^3 + c_4e^4 + c_5e^5 + c_6e^6 + c_7e^7 + c_8e^8 + O(e^9)) \tag{2.6}$$

and

$$f'(x) = f'(x^*)(1 + 2c_2e + 3c_3e^2 + 4c_4e^3 + 5c_5e^4 + 6c_6e^5 + 7c_7e^6 + 8c_8e^7 + 9c_9e^8 + O(e^9)). \tag{2.7}$$

Thus,

$$\begin{aligned} \tilde{e} &= c_2e^2 + (2c_3 - 2c_2^2)e^3 + (-7c_2c_3 + 4c_2^3 + 3c_4)e^4 + (-8c_2^4 + 20c_2^2c_3 - 6c_3^2 - 10c_2c_4 + 4c_5)e^5 \\ &\quad + (16c_2^5 - 52c_2^3c_3 + 28c_2^2c_4 - 17c_3c_4 + c_2(33c_3^2 - 13c_5) + 5c_6)e^6 \\ &\quad - 2(16c_2^6 - 64c_2^4c_3 - 9c_3^3 + 36c_2^3c_4 + 6c_4^2 + 9c_2^2(7c_3^2 - 2c_5) + 11c_3c_5 + c_2(-46c_3c_4 + 8c_6) - 3c_7)e^7 \\ &\quad + (64c_2^7 - 304c_2^5c_3 + 176c_2^4c_4 + 75c_3^2c_4 + c_2^3(408c_3^2 - 92c_5) - 31c_4c_5 - 27c_3c_6 \\ &\quad + c_2^2(-348c_3c_4 + 44c_6) + c_2(-135c_3^3 + 64c_4^2 + 118c_3c_5 - 19c_7) + 7c_8)e^8 + \dots. \end{aligned} \tag{2.8}$$

Using Taylor's approach, we may expand $f(\psi_{NR_2}(x))$ about $x^*$ and obtain

$$f(\psi_{NR_2}(x)) = f'(x^*)(\tilde{e} + c_2\tilde{e}^2 + c_3\tilde{e}^3 + c_4\tilde{e}^4 + O(\tilde{e}^5)). \tag{2.9}$$

We obtain by simplifying and substituting these equations (2.6)-(2.8) and (2.4) in the (2.3),

$$\psi_{SSDM_4}(x) - x^* = \left(\left(5 - \frac{H''(0)}{2}\right)c_2^3 - c_2c_3\right)e^4 + O(e^5).$$

This shows that fourth-order convergence is achieved by the suggested classes of approaches. □

We are able to generate a new optimal fourth order method in (2.4) by selecting any random value for $H''(1)$. Selecting $H''(1) = 2$ yields new suggested approaches as follows:

$$\left.\begin{aligned} &\psi_{SSDM_4}(x)=\psi_{NR_2}(x)-H(\tau)\frac{f(\psi_{NR_2}(x))}{f'(x)},\\ &H(\tau)=1-2(\tau-1)+(\tau-1)^2 \text{and} \tau=1-\frac{f(\psi_{NR_2}(x))}{f(x)}. \end{aligned}\right\} \tag{2.10}$$

This method (2.10) has the following error equation $\psi_{SSDM_4}(x)-x^*=(4c_2^3-c_2c_3))e^4+O(e^5)$. $EI_{SSDM_4}=1.587$ is the efficiency of the method (2.15).

## 2.2 An Eighth-Order Optimum Technique

Next, we try the following method to obtain a new optimal eighth order *IF*,

$$\psi_{SSDM_8}(x)=\psi_{SSDM_4}(x)-\frac{f(\psi_{SSDM_4}(x))}{f'(\psi_{SSDM_4}(x))}.$$

With five function evaluations, the aforementioned one exhibits eighth order convergence. However, this is not the best approach. In order to estimate $f'(\psi_{SSDM_4}(x))$, we must minimise a function while maintaining the same convergence order. This polynomial is used to estimate the optimal,

$$q(t)=b_3(t-x)^3+b_2(t-x)^2+b_1(t-x)+b_0, \tag{2.11}$$

which fulfills

$$q'(x)=f'(x),\quad q(x)=f(x),\quad q(\psi_{NR_2}(x))=f(\psi_{NR_2}(x)),\quad q(\psi_{SSDM_4}(x))=f(\psi_{SSDM_4}(x)).$$

When the aforementioned requirements are applied to (2.11), there are generated four linear equations: $b_0$, $b_1$, $b_2$, and $b_3$. $b_0=f(x)$ and $b_1=f'(x)$ follow from $q(x)=f(x)$, $q'(x)=f'(x)$. $b_2$ and $b_3$ are found by solving these equations:

$$f(\psi_{NR_2}(x))=b_3(\psi_{NR_2}(x)-x)^3+b_2(\psi_{NR_2}(x)-x)^2+f'(x)(\psi_{NR_2}(x)-x)+f(x),$$
$$f(\psi_{SSDM_4}(x))=b_3(\psi_{SSDM_4}(x)-x)^3+b_2(\psi_{SSDM_4}(x)-x)^2+f'(x)(\psi_{SSDM_4}(x)-x)+f(x).$$

Therefore, by using divided differences, the aforementioned equations become simpler to

$$f[\psi_{NR_2}(x),x,x]=b_2+b_3(\psi_{NR_2}(x)-x), \tag{2.12}$$
$$f[\psi_{SSDM_4}(x),x,x]=b_2+b_3(\psi_{SSDM_4}(x)-x). \tag{2.13}$$

Equations (2.12) and (2.13) can be solved to yield

$$\left.\begin{aligned} b_2&=\frac{f[\psi_{NR_2}(x),x,x](\psi_{SSDM_4}(x)-x)-f[\psi_{SSDM_4}(x),x,x](\psi_{NR_2}(x)-x)}{\psi_{SSDM_4}(x)-\psi_{NR_2}(x)},\\ b_3&=\frac{f[\psi_{SSDM_4}(x),x,x]-f[\psi_{NR_2}(x),x,x]}{\psi_{SSDM_4}(x)-\psi_{NR_2}(x)}. \end{aligned}\right\} \tag{2.14}$$

Furthermore, we have the estimation using eq. (2.14),

$$f'(\psi_{SSDM_4}(x))\approx q'(\psi_{SSDM_4}(x))=b_1+2b_2(\psi_{SSDM_4}(x)-x)+3b_3(\psi_{SSDM_4}(x)-x)^2.$$

Lastly, we provide a fresh, eighth-order optimum technique as

$$\psi_{SSDM_8}(x)=\psi_{SSDM_4}(x)-\frac{f(\psi_{SSDM_4}(x))}{f'(x)+2b_2(\psi_{SSDM_4}(x)-x)+3b_3(\psi_{SSDM_4}(x)-x)^2}. \tag{2.15}$$

$EI_{SSDM_8}=1.682$ is the efficiency of the approach (2.15).

We use MATHEMATICA software to demonstrate the convergence analysis of the suggested *IF*s (2.15).

**Theorem 2.2.** *Assume that the function $f : D \subset \mathbb{R} \to \mathbb{R}$ is sufficiently smooth and has derivatives that are continuous. When $x_0$ is selected within a suitably small neighbourhood of $x^*$ and $f(x)$ has a simple root $x^*$ in the open interval $D$, the* (2.15) *is of eighth order convergence and fulfils the error equation:*

$$\psi_{SSDM_8}(x) - x^* = c_2^2(4c_2^2 - c_3)(4c_2^3 - c_2c_3 + c_4)e^8 + O(e^9). \tag{2.16}$$

## 3. Numerical Examples

We will test a number of cases to demonstrate the effectiveness of the new optimal schemes, $SSDM_4$ and $SSDM_8$. We compare the new schemes with the optimal fourth-order methods $SB_4$ presented by Sharma and Bahl [21], $CM_4$ proposed by Chun *et al*. [6], $SJ_4$ presented by Singh and Jaiswal [22], and optimal eighth order methods $KT_8$ proposed by Kung and Traub [14], $LW_8$ presented by Liu and Wang [15], $PNPD_8$ developed by Petkovic *et al*. [18], $SA_8$ proposed by Sharma and Arora [20], $CFGT_8$ presented by Cordero *et al*. [7], $CTV_8$ developed by Cordero *et al*. [9].

500 significant digits have been used in numerical calculations performed in the MATLAB program. The halting criteria for the iterative process meeting $error = |x_N - x_{N-1}| < \epsilon$, where the number of iterations required for convergence is $N$ and $\epsilon = 10^{-50}$, has been applied. The order of convergence in computing is provided by (Cordero and Torregrosa [8]),

$$\rho = \frac{\ln|(x_N - x_{N-1})/(x_{N-1} - x_{N-2})|}{\ln|(x_{N-1} - x_{N-2})/(x_{N-2} - x_{N-3})|}.$$

Below are the test functions for our investigation along with their simple zeros:

$$f_1(x) = \sin(2\cos x) - 1 - x^2 + e^{\sin(x^3)}, \qquad x^* = -0.7848259876612125352\ldots,$$

$$f_2(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5, \qquad x^* = -1.2076473271309189270\ldots,$$

$$f_3(x) = x^3 + 4x^2 - 10, \qquad x^* = 1.3652100134140968457\ldots,$$

$$f_4(x) = \sin(x) + \cos(x) + x, \qquad x^* = -0.4566447045676308244\ldots,$$

$$f_5(x) = \frac{x}{2} - \sin x, \qquad x^* = 1.8953942670339809471\ldots,$$

$$f_6(x) = x^2 + \sin\left(\frac{x}{5}\right) - \frac{1}{4}, \qquad x^* = 0.4099220179891371316\ldots.$$

The equivalent results for $f_1 - f_6$ are displayed in Table 1. When compared to other methods, we find that the suggested method $SSDM_4$ converges with the least amount of error and in fewer or equivalent iterations. Take note that in the $f_5$ function, the $SB_4$ and $SJ_4$ techniques are diverging. As a result, the suggested approach $SSDM_4$ can be regarded as sufficiently competent in comparison to other comparable present methods.

Additionally, the related results for $f_1 - f_6$ are displayed in Tables 2-4. The theoretical order and the computational order of convergence coincide for all examined functions. It is seen that the $8^{th}PNPD$ technique is diverging in the function $f_5$, whereas the suggested method is convergent with fewer iterations and minimal error. The suggested approach is generally convergent with fewer iterations and a lower *total number of functional evaluations* (TNFE) with the least amount of error. As a result, the suggested approach $SSDM_8$ can be regarded as sufficiently competent in comparison to other similar current methods.

**Table 1.** Comparison of numerical outcomes

| | $N$ | TNFE | $\lvert x_N - x_{N-1}\rvert$ | $\rho$ | $N$ | TNFE | $\lvert x_N - x_{N-1}\rvert$ | $\rho$ |
|---|---|---|---|---|---|---|---|---|
| Methods | | $f_1(x)$, $x_0 = -0.9$ | | | | $f_2(x)$, $x_0 = -1.6$ | | |
| $NR_2$ | 7 | 14 | 7.7336e-74 | 1.99 | 9 | 18 | 9.2727e-74 | 1.99 |
| $SB_4$ | 4 | 12 | 9.7235e-64 | 3.99 | 5 | 15 | 1.4267e-65 | 3.99 |
| $CM_4$ | 4 | 12 | 1.4696e-64 | 3.99 | 5 | 15 | 1.1070e-72 | 3.99 |
| $SJ_4$ | 4 | 12 | 3.0633e-62 | 3.99 | 5 | 15 | 9.9781e-56 | 3.99 |
| $SSDM_4$ | 4 | 12 | 6.0046e-71 | 3.99 | 5 | 15 | 7.29140e-129 | 4.00 |
| Methods | | $f_3(x)$, $x_0 = 0.9$ | | | | $f_4(x)$, $x_0 = -1.9$ | | |
| $NR_2$ | 8 | 16 | 1.3534e-72 | 2.00 | 8 | 16 | 1.6062e-72 | 1.99 |
| $SB_4$ | 5 | 15 | 4.5742e-106 | 3.99 | 5 | 15 | 6.0481e-92 | 3.99 |
| $CM_4$ | 5 | 15 | 4.7335e-108 | 3.99 | 5 | 15 | 2.7342e-93 | 3.99 |
| $SJ_4$ | 5 | 15 | 3.0354e-135 | 3.99 | 5 | 15 | 9.5023e-95 | 3.99 |
| $SSDM_4$ | 5 | 15 | 2.6336e-166 | 3.99 | 5 | 15 | 1.4403e-112 | 3.99 |
| Methods | | $f_5(x)$, $x_0 = 1.2$ | | | | $f_6(x)$, $x_0 = 0.8$ | | |
| $NR_2$ | 9 | 18 | 1.3564e-83 | 1.99 | 8 | 16 | 3.2034e-72 | 1.99 |
| $SB_4$ | | | **Diverge** | | 5 | 15 | 2.8269e-122 | 3.99 |
| $CM_4$ | 14 | 42 | 6.8660e-134 | 3.99 | 5 | 15 | 7.8638e-127 | 3.99 |
| $SJ_4$ | | | **Diverge** | | 5 | 15 | 1.4355e-114 | 3.99 |
| $SSDM_4$ | 6 | 18 | 2.3555e-152 | 3.99 | 5 | 15 | 1.1419e-159 | 3.99 |

**Table 2.** Comparison of numerical outcomes

| | $N$ | TNFE | $\lvert x_N - x_{N-1}\rvert$ | $\rho$ | $N$ | TNFE | $\lvert x_N - x_{N-1}\rvert$ | $\rho$ |
|---|---|---|---|---|---|---|---|---|
| Methods | | $f_1(x)$, $x_0 = -0.9$ | | | | $f_2(x)$, $x_0 = -1.6$ | | |
| $KT_8$ | 3 | 12 | 1.6238e-61 | 7.91 | 4 | 16 | 7.2890e-137 | 7.99 |
| $LW_8$ | 3 | 12 | 4.5242e-59 | 7.91 | 4 | 16 | 1.1195e-170 | 8.00 |
| $PNPD_8$ | 3 | 12 | 8.8549e-56 | 7.87 | 4 | 16 | 2.3461e-85 | 7.99 |
| $SA_8$ | 3 | 12 | 3.4432e-60 | 7.88 | 4 | 16 | 8.4343e-121 | 8.00 |
| $CFGT_8$ | 3 | 12 | 1.1715e-82 | 7.77 | 5 | 16 | 2.0650e-183 | 7.99 |
| $CTV_8$ | 3 | 12 | 4.4923e-61 | 7.94 | 4 | 16 | 2.3865e-252 | 7.99 |
| $SSDM_8$ | 3 | 12 | 1.1416e-96 | 7.96 | 4 | 16 | 8.9301e-269 | 8.00 |

**Table 3.** Comparison of numerical outcomes

| | $N$ | TNFE | $\lvert x_N - x_{N-1}\rvert$ | $\rho$ | $N$ | TNFE | $\lvert x_N - x_{N-1}\rvert$ | $\rho$ |
|---|---|---|---|---|---|---|---|---|
| Methods | $f_3(x)$, $x_0 = 0.9$ | | | | $f_4(x)$, $x_0 = -1.9$ | | | |
| $KT_8$ | 4 | 16 | 5.0765e-216 | 7.99 | 4 | 16 | 5.5095e-204 | 8.00 |
| $LW_8$ | 4 | 16 | 2.7346e-213 | 7.99 | 4 | 16 | 3.7210e-146 | 8.00 |
| $PNPD_8$ | 4 | 16 | 9.9119e-71 | 8.02 | 4 | 16 | 2.0603e-116 | 7.98 |
| $SA_8$ | 4 | 16 | 1.5396e-122 | 8.00 | 4 | 16 | 2.2735e-136 | 7.99 |
| $CFGT_8$ | 4 | 16 | 2.4091e-260 | 7.99 | 4 | 16 | 4.7007e-224 | 7.99 |
| $CTV_8$ | 4 | 16 | 3.8782e-288 | 8.00 | 4 | 16 | 3.7790e-117 | 7.99 |
| $SSDM_8$ | 4 | 16 | 3.5460e-319 | 7.99 | 4 | 16 | 2.9317e-235 | 7.99 |

**Table 4.** Comparison of numerical outcomes

| | $N$ | TNFE | $\lvert x_N - x_{N-1}\rvert$ | $\rho$ | $N$ | TNFE | $\lvert x_N - x_{N-1}\rvert$ | $\rho$ |
|---|---|---|---|---|---|---|---|---|
| Methods | $f_5(x)$, $x_0 = 1.2$ | | | | $f_6(x)$, $x_0 = 0.8$ | | | |
| $KT_8$ | 5 | 20 | 2.6836e-182 | 7.99 | 4 | 16 | 6.0701e-234 | 7.99 |
| $LW_8$ | 6 | 24 | 4.6640e-161 | 7.99 | 4 | 16 | 6.1410e-228 | 7.99 |
| $PNPD_8$ | | | **Diverge** | | 4 | 16 | 3.6051e-190 | 7.99 |
| $SA_8$ | 7 | 32 | 2.1076e-215 | 9.00 | 4 | 16 | 5.9608e-245 | 8.00 |
| $CFGT_8$ | 5 | 20 | 0 | 7.99 | 4 | 16 | 1.0314e-232 | 7.99 |
| $CTV_8$ | 5 | 20 | 1.6474e-219 | 9.00 | 4 | 16 | 1.0314e-274 | 8.00 |
| $SSDM_8$ | 4 | 16 | 1.3183e-98 | 7.98 | 4 | 16 | 1.2160e-296 | 7.99 |

## 4. Applications to Projectile Motion Problem

The classical projectile problem is examined by Babajee and Madhu [5], and Kantrowitz and Neumann [13], where a projectile is launched onto a hill at an angle $\theta$ relative to the horizontal and from a tower of height $h > 0$. The impact function, defined by the function $\omega$, is dependent on the horizontal distance, $x$. The ideal launch angle $\theta_m$ that maximises the horizontal distance is what we are looking for. We do not account for air resistances in our calculations. The projectile's motion is described by the path function $y = P(x)$, which is provided by

$$P(x) = h + x\tan\theta - \frac{gx^2}{2v^2}\sec^2\theta \tag{4.1}$$

Following the projectile's impact with the hill, $P(x) = \omega(x)$ for a given value $x$. Finding $\theta$ at a value that maximises $x$ is our goal,

$$\omega(x) = P(x) = h + x\tan\theta - \frac{gx^2}{2v^2}\sec^2\theta \tag{4.2}$$

By implicitly differentiating equation (4.2) with respect to $\theta$, we obtain

$$\omega'(x)\frac{dx}{d\theta} = x\sec^2\theta + \frac{dx}{d\theta}\tan\theta - \frac{g}{v^2}\left(x^2\sec^2\theta\tan\theta + x\frac{dx}{d\theta}\sec^2\theta\right) \tag{4.3}$$

Setting $\frac{dx}{d\theta} = 0$ in eq. (4.3), we have

$$x_m = \frac{v^2}{g}\cot\theta_m \tag{4.4}$$

or

$$\theta_m = \arctan\left(\frac{v^2}{g\,x_m}\right) \tag{4.5}$$

A path that encompasses and intersects every feasible path is known as an encompassing parabola. By maximising the projectile's height for a given horizontal distance $x$, Henelsmith[1] constructed an enveloping parabola, which will yield the path that encloses all potential trajectories. Let $w = \tan\theta$, then eq. (4.1) becomes

$$y = P(x) = h + xw - \frac{gx^2}{2v^2}(1+w^2). \tag{4.6}$$

Using $y' = 0$ and differentiating eq. (4.6) with respect to $w$, Henelsmith obtained

$$\left.\begin{aligned} y' &= x - \frac{xg^2}{v^2}(w) = 0,\\ w &= \frac{v^2}{gx}, \end{aligned}\right\} \tag{4.7}$$

so that the enveloping parabola defined by

$$y_m = \rho(x) = h + \frac{v^2}{2g} - \frac{gx^2}{2v^2}. \tag{4.8}$$

Identifying $x_m$ that fulfills the equation $\rho(x) = \omega(x)$ and calculating $\theta_m$ using eq. (4.5) are the first steps in solving the projectile problem because we need to determine the point on the enveloping parabola $\rho$ where it intersects the impact function $\omega$. Next, we need to determine the value of $\theta$ that, on the surrounding parabola, corresponds to this point. With $h = 10$ and $v = 20$, we select an impact function that is linear $\omega(x) = 0.4x$. Let $g = 9.8$. The non-linear equation is then solved by using our $IF$s beginning at $x_0 = 30$,

$$f(x) = \rho(x) - \omega(x) = h + \frac{v^2}{2g} - \frac{gx^2}{2v^2} - 0.4x,$$

whose root is given by $x_m = 36.102990117\ldots$ and

$$\theta_m = \arctan\left(\frac{v^2}{g\,x_m}\right) = 48.5^\circ.$$

The proposed approach $SSDM_8$ is converging more effectively than the other compared methods, as Table 5 demonstrates. Furthermore, we note that the theoretical order of convergence and the computational order of convergence coincide.

[1]N. Henelsmith, *Projectile Motion: Finding the Optimal Launch Angle*, Whitman College, Washington, USA, 38 pages (2016), URL: https://www.whitman.edu/Documents/Academics/Mathematics/2016/Henelsmith.pdf.

**Table 5.** Projectile problem outcomes

| *IF* | *N* | error | cpu time(s) | $\rho$ |
|---|---|---|---|---|
| $NR_2$ | 7 | 4.3980e-76 | 1.074036 | 1.99 |
| $SSDM_4$ | 4 | 4.3980e-76 | 0.902015 | 3.99 |
| $KT_8$ | 3 | 1.5610e-66 | 0.658235 | 8.03 |
| $LW_8$ | 3 | 7.8416e-66 | 0.672524 | 8.03 |
| $PNPD_8$ | 3 | 4.2702e-57 | 0.672042 | 8.05 |
| $SA_8$ | 3 | 1.2092e-61 | 0.654623 | 8.06 |
| $CTV_8$ | 3 | 3.5871e-73 | 0.689627 | 8.02 |
| $SSDM_8$ | 3 | 4.3980e-80 | 0.513142 | 8.02 |

## 5. Basins of Attraction

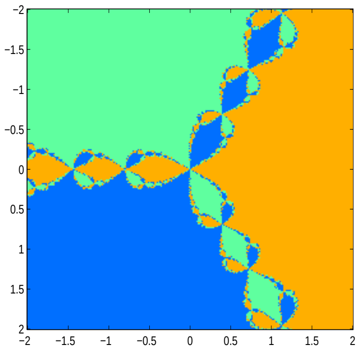
Analysing the rational function's dynamic behaviour in relation to an iterative process provides valuable insights into the method's stability and convergence. Amat *et al.* [4] and Scott *et al.* [19] provide fundamental definitions and dynamic notions of rational functions.

Applying our iterative methods, we pick a square with $256 \times 256$ points that is $\mathbb{R} \times \mathbb{R} = [-2,2] \times [-2,2]$. We start in every $z^{(0)}$ in the square. If, for a maximum of 100 iterations, the sequence generated by the iterative technique attempts a zero $z_j^*$ of the polynomial with a tolerance $|f(z^{(k)})| < 1e-4$, we conclude that $z^{(0)}$ is in the basin of attraction of this zero. We label this point $z^{(0)}$ with colours if $|z^{(N)} - z_j^*| < 1e-4$. This is done if the iterative technique, it begins in $z^{(0)}$ and, in $N$ iterations ($N \leq 100$), reaches a zero. We determine that the starting point has diverged if $N > 50$, and we apply a dark blue colour. The following describes the basins of attraction for the Newton's method and a few higher order Newton-type methods for finding the complex roots of the polynomials $p_1(z) = z^3 - 1$ and $p_2(z) = z^5 - 1$.

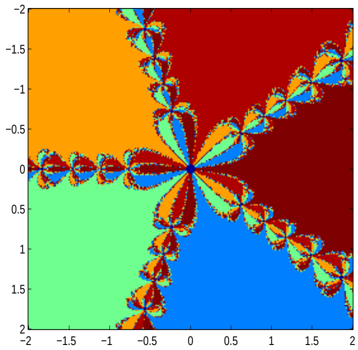
Figure 1 displays the polynomiographs for the approaches to the polynomials $p_1(z)$ and $p_2(z)$ for the $NR_2$. The polynomiographs for the fourth order iterative approaches for the polynomial $p_1(z)$ are displayed in Figure 2. The polynomiographs for the ninth order iterative approaches for the polynomial $p_1(z)$ are displayed in Figure 3. The polynomiographs for the fourth order iterative approaches for the polynomial $p_2(z)$ are displayed in Figure 4. The polynomiographs for the ninth order iterative approaches for the polynomial $p_2(z)$ are displayed in Figure 5.

It is noted that the performance of the approaches $NR_2$, $SSDM_4$, and $SSDM_8$ is remarkable in the $p_1(z)$. In close proximity of the boundary points, the methods $SB_4$, $KT_8$, and $LW_8$, exhibit some chaotic behaviour. In this scenario, the approaches $CM_4$, $SJ_4$, $PNPD_8$, $SA_8$, and $CFGT_8$ are sensitive to the initial guess selection.

Also note that the approaches $SSDM_4$ and $SSDM_8$ exhibit some chaotic behaviour in the vicinity of the boundary points for $p_2(z)$. $NR_2$, $SB_4$, $CM_4$, and $SJ_4$ are the techniques $KT_8$, In this instance, the values of $LW_8$, $PNPD_8$, $SA_8$, and $CFGT_8$ are all sensitive to the initial guess made.

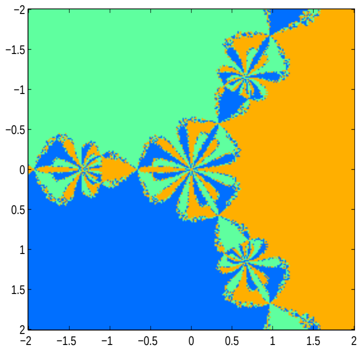


$NR_2$ for $p_1(z) = z^3 - 1$

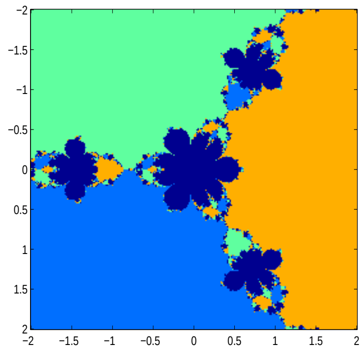


$NR_2$ for $p_2(z) = z^5 - 1$

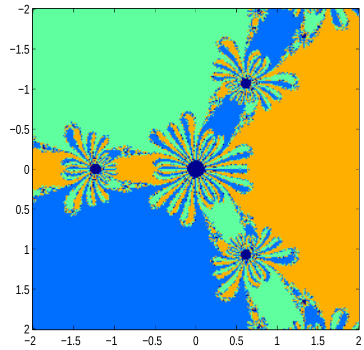
**Figure 1.** Newton's technique ($NR_2$) basins of attraction



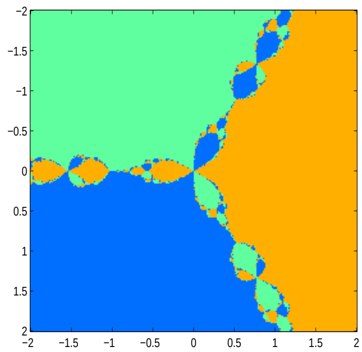
$SB_4$



$CM_4$

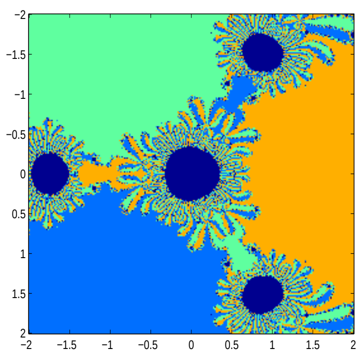


$SJ_4$

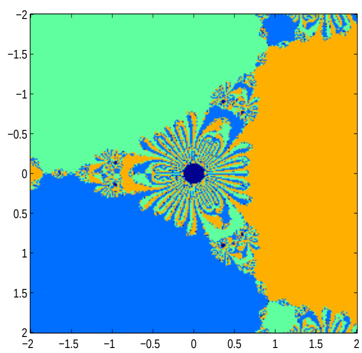


$SSDM_4$

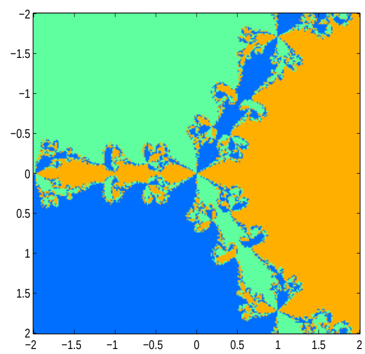
**Figure 2.** $p_1(z) = z^3 - 1$ basins of attraction



$PNPD_8$

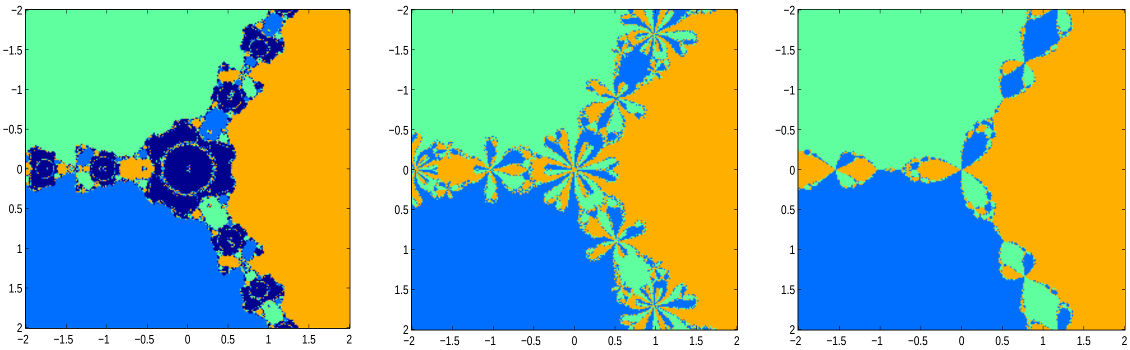


$SA_8$



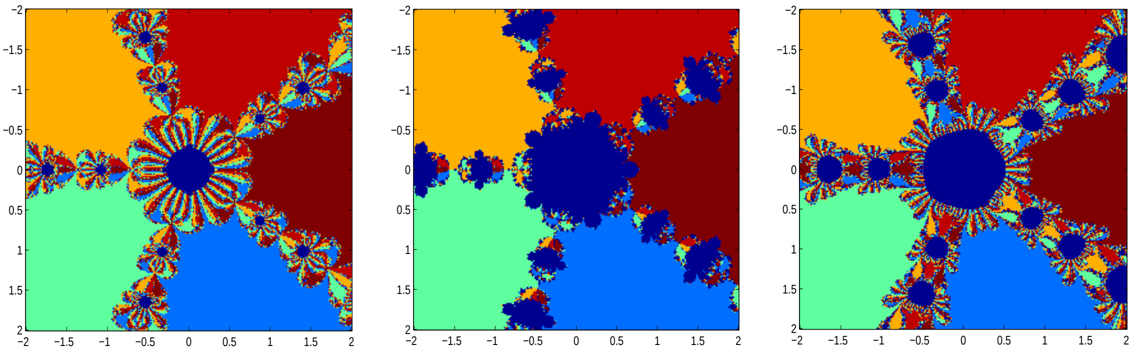
$KT_8$

Figure *Contd.*

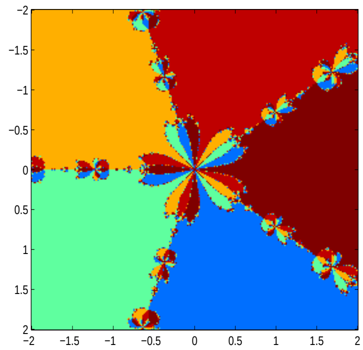


$CFGT_8$ $LW_8$ $SSDM_8$

**Figure 3.** $p_1(z) = z^3 - 1$ basins of attraction

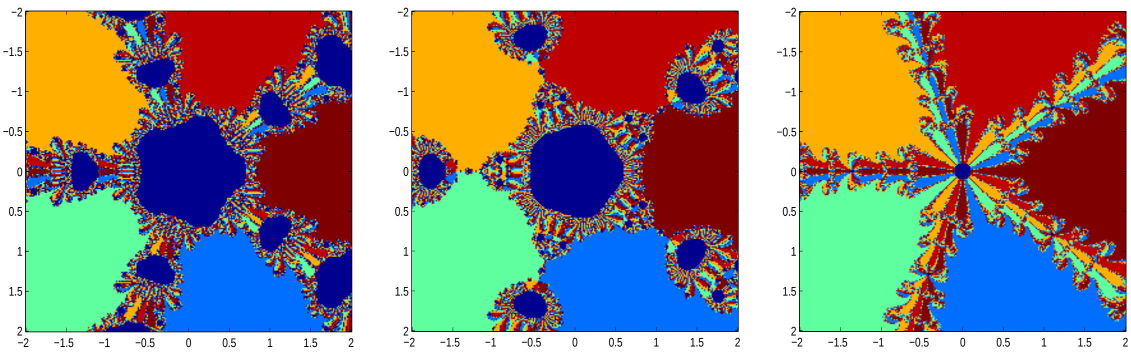


$SB_4$ $CM_4$ $SJ_4$



$SSDM_4$

**Figure 4.** $p_1(z) = z^5 - 1$ basins of attraction



$PNPD_8$ $SA_8$ $KT_8$

Figure *Contd*.

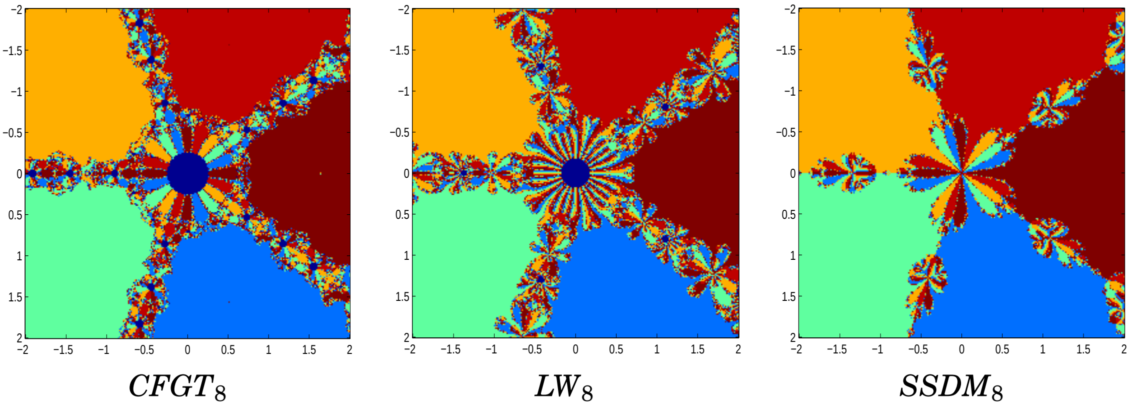


**Figure 5.** $p_1(z) = z^5 - 1$ basins of attraction

## 6. Conclusion

In this work, we established a family of iterative algorithms for solving nonlinear equations that is optimal at the fourth and eighth orders, respectively. Three and four function evaluations are needed for the approach to obtain an order of convergence of four and eight, respectively. The Kung-Traub conjecture is met in the sense of convergence analysis and numerical examples. To demonstrate the superiority of the proposed methods $SSDM_4$ and $SSDM_8$, we have tested few examples with existing recognised methods. According to the results of the numerical results, the new methods could be a useful alternative for solving nonlinear equations. Additionally, we address a real-world application to demonstrate the efficacy of the proposed methods. By displaying their corresponding fractals, more research has been done on the complex plane to uncover the basins of attraction of such approaches for solving nonlinear equations.

### Competing Interests

The authors declare that they have no competing interests.

### Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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