A Characterisation of Strong Integer Additive Set-Indexers of Graphs

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Abstract. Let $\mathbb{N}_0$ be the set of all non-negative integers and $\mathcal{P}(\mathbb{N}_0)$ be its power set. An integer additive set-indexer (IASI) of a given graph $G$ is an injective function $f : V(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$ such that the induced function $f^+ : E(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$ defined by $f^+(uv) = f(u) + f(v)$ is also injective, where $f(u) + f(v)$ is the sum set of $f(u)$ and $f(v)$. If $f^+(uv) = k \forall uv \in E(G)$, then $f$ is said to be a $k$-uniform IASI. An IASI $f$ is said to be a strong IASI if $|f^+(uv)| = |f(u)||f(v)| \forall uv \in E(G)$. In this paper, we study the characteristics of certain graph classes, graph operations and graph products that admit strong integer additive set-indexers.

Keywords. Set-indexer; Integer additive set-indexer; Strong integer additive set-indexer; Difference set; Nourishing number of a graph

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1. Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [4], [9], [15] and [20]. For graph products, we further refer to [14] and [16]. Unless mentioned otherwise, all graphs considered here are simple, finite and have no isolated vertices.

The sumset of two non-empty sets $A$ and $B$, denoted by $A + B$, is the set defined by $A + B = \{a + b : a \in A, b \in B\}$. If either $A$ or $B$ is countably infinite, then $A + B$ will also be countably infinite. Hence, all the sets in present discussion will be non-empty finite sets. We
denote the cardinality of a set \( A \) by \(|A|\).

Invoking the concepts of sumsets of finite sets, the notion of an integer additive set-indexer of a graph \( G \) is introduced as follows.

**Definition 1.1** \((17)\). An integer additive set-indexer (IASI) of a given graph \( G \) is defined as an injective function \( f : V(G) \rightarrow \mathcal{P}(\mathbb{N}_0) \) such that the induced function \( f^+ : E(G) \rightarrow \mathcal{P}(\mathbb{N}_0) \) defined by \( f^+(uv) = f(u) + f(v) \) is also injective. A graph \( G \) which admits an integer additive set-indexer is called an integer additive set-indexed graph (IASI-graph).

The cardinality of the set-label of an element (a vertex or an edge) of an IASI-graph is called the set-indexing number of that element.

The following theorem establishes the bounds for the cardinality of the sumset of two non-empty sets of integers.

**Theorem 1.2** \((17)\). Let \( A \) and \( B \) be two non-empty sets of integers. Then, \( |A| + |B| - 1 \leq |A + B| \leq |A| |B| \).

The IASIs of a given graph \( G \) with respect to which all the edges of \( G \) have the highest possible set-indexing number are of special interest. Hence, we have introduced the following notion.

**Definition 1.3** \((19)\). If a graph \( G \) has a set-indexer \( f \) such that \(|f^+(uv)| = |f(u) + f(v)| = |f(u)||f(v)| \) for all vertices \( u \) and \( v \) of \( G \), then \( f \) is said to be a strong IASI of \( G \). A graph which admits a strong IASI is called a strong IASI-graph.

**Definition 1.4** \((19)\). If \( G \) is a graph which admits a \( k \)-uniform IASI and \( V(G) \) is \( l \)-uniformly set-indexed, then \( G \) is said to have a \((k,l)\)-completely uniform IASI or simply a completely uniform IASI.

We use the notation \( A < B \) in the sense that \( A \cap B = \emptyset \). We notice that the relation \( < \) is symmetric, but need not be reflexive and transitive. By the sequence \( A_1 < A_2 < A_3 < \ldots < A_n \), we mean that the given sets are pairwise disjoint.

The difference set of a non-empty set \( A \), denoted by \( D_A \), is the set defined by \( D_A = \{|a - b| : a, b \in A\} \). The following lemma provides a necessary and sufficient condition for the sumset of two sets to have the highest possible cardinality.

**Lemma 1.5** \((19)\). Let \( A, B \) be two non-empty subsets of \( \mathbb{N}_0 \). Then, \(|A + B| = |A||B|\) if and only if their difference sets, denoted by \( D_A \) and \( D_B \) respectively, follow the relation \( D_A < D_B \). In other words, \(|A + B| = |A||B|\) if and only if \( D_A \cap D_B = \emptyset \).

A necessary and sufficient condition for a given complete graph to admit a strong IASI is given below.

**Theorem 1.6** \((19)\). Let each vertex \( v_i \) of the complete graph \( K_n \) be labeled by the set \( A_i \in \mathcal{P}(\mathbb{N}_0) \). Then \( K_n \) admits a strong IASI if and only if there exists a finite sequence of sets \( D_1 < D_2 < D_3 < \cdots < D_n \) where each \( D_i \) is the set of all differences between any two elements of the set \( A_i \).
The hereditary nature of the existence of strong IASI of a given graph has been established in the following result.

**Theorem 1.7** ([19]). If a graph \( G \) admits a strong IASI then its subgraphs also admit strong IASI.

**Corollary 1.8** ([19]). A connected graph \( G \) (on \( n \) vertices) admits a strong IASI if and only if each vertex \( v_i \) of \( G \) is labeled by a set \( A_i \) in \( \mathcal{P}(\mathbb{N}_0) \) and there exists a finite sequence of sets \( D_1 < D_2 < D_3 < \cdots < D_m \), where \( m \leq n \) is a positive integer and each \( D_i \) is the set of all differences between any two elements of the set \( A_i \).

### 2. New Results on Strong IASI Graphs

In this paper, as an extension to the studies on strong IASI-graphs, done in [19], we study certain characteristics and properties of certain graph classes, graph operations and graph products which admit strong IASIs.

#### 2.1 Strong IASI of Certain Graph Classes

Let us denote the difference set of the set-label \( A_i \) of a vertex \( v_i \) in \( G \) by \( D_i \). The relation \(<\) is called the difference relation between the set-labels of the corresponding vertices in \( G \). A chain of difference sets is the sequence of difference sets \( D_1 < D_2 < D_3 < \cdots < D_n \) and the length of a chain is the number of difference sets in that chain.

For any strong IASI-graph with \( m \) vertices and \( n \) edges, there are \( m \) difference sets, one each for each vertex and \( n \) relations \(<\), one each corresponding to each edge of \( G \). Note that all these difference sets need not be necessarily pair wise disjoint. But, if \( G \) is a strong IASI-graph, then the difference sets of two adjacent vertices \( u \) and \( v \) in \( G \) must be disjoint. That is, there exists a difference relation for the set-labels of any two adjacent vertices in \( G \) and these relations forms one or more chain of difference sets. The lengths of such chains of difference sets are noteworthy and hence we have the following notion.

**Definition 2.1.** The nourishing number of a set-labeled graph is the minimum length of the maximal chain of difference sets in \( G \). The nourishing number of a graph \( G \) is denoted by \( \kappa(G) \).

**Remark 2.2.** In other words, the nourishing number of a given strong IASI-graph can be defined as the minimum value required for the order of its maximal (connected) subgraph, the difference sets of the set-labels of all whose vertices are pair wise disjoint.

The study about nourishing number of different graphs and graph classes arouses much interest. In this section, we discuss about the nourishing number of various graphs.

In view of the above definition we can rewrite Theorem 1.6, as given below.

**Theorem 2.3.** The nourishing number of a complete graph is \( n \). That is, \( \kappa(K_n) = n \).

The following theorem determines the nourishing number of bipartite graphs.

**Theorem 2.4.** The nourishing number of a bipartite graph is 2.
Theorem 2.7. Let \( G \) be a bipartite graph with bipartition \((X, Y)\) which admits a strong IASI. Let \( X = \{u_1, u_2, u_3, \ldots, u_m\} \) and \( Y = \{v_1, v_2, v_3, \ldots, v_n\} \). Define \( f : V(G) \to \mathcal{P}(\mathbb{N}_0) \) such that for two adjacent vertices \( u_i \in X \) and \( v_j \in Y \), the difference sets of \( f(u_i) \) and \( f(v_j) \), denoted by \( D_{u_i} \) and \( D_{v_j} \) respectively, hold the relation \( D_{u_i} < D_{v_j} \). Hence, corresponding to each edge in \( G \), there exists a difference relation between the set-labels of its end vertices. Hence, \( f \) is a strong IASI and the minimum length of the maximal chain in \( G \) is 2. That is, \( \kappa(G) = 2 \).

\[ \square \]

Proposition 2.5. The nourishing number of a triangle-free graph is 2.

Proof. Since \( G \) is a triangle-free graph, \( G \) can not be a complete graph. Since the relation \( < \) is not transitive, we can not find a chain consisting of three or more difference sets corresponding to the set-labels of vertices of \( G \). Hence \( \kappa(G) = 2 \). \[ \square \]

Due to Proposition 2.5, our problem of finding the nourishing number of a graph \( G \) is reduced to finding the order of a maximal complete subgraph of \( G \). We know that a clique of graph \( G \) is a complete subgraph of \( G \). We recall that the clique number \( \omega(G) \) of a graph is the number of vertices in a maximal clique in \( G \). Hence, we have

Proposition 2.6. The nourishing number of a graph \( G \) is equal to the clique number \( \omega \) of \( G \).

Proof. Let \( H \) be the maximal clique of a given graph \( G \). Then, \( H \) is complete graph. Hence, by Theorem [1.6], there exists a chain of difference sets \( D_1 < D_2 < D_3 < \ldots < D_r \), where \( r = |V(H)| \). Moreover, no cliques in \( G - H \) can have more vertices in \( G \) than \( H \). Therefore, \( \kappa(G) = r = \omega(G) \). \[ \square \]

Interesting questions that arise in this context are about the nourishing number of different graph operations. In the following discussions, we address these problems. First, we check the admissibility of strong IASIs by the union of two graphs and its nourishing number in the following results.

2.2 Strong IASIs of Graph Operations

By the term last vertex of a subgraph \( H \) of \( G \), we mean a vertex \( v \in V(H) \) whose adjacent vertices in \( G \) are not in \( V(H) \).

Theorem 2.7. The union \( G_1 \cup G_2 \) of two graphs \( G_1 \) and \( G_2 \), admits a strong IASI if and only if both \( G_1 \) and \( G_2 \) admit strong IASIs.

Proof. Let \( G_1 \) and \( G_2 \) admit strong IASIs say \( f_1 \) and \( f_2 \). If \( G_1 \) and \( G_2 \) are disjoint graphs, then we observe that the IASI \( f \) defined by \( f(v) = f_i(v) \) if \( v \in V(G_i), i = 1, 2 \) is a strong IASI for \( G_1 \cup G_2 \).

If \( G_1 \) and \( G_2 \) are not disjoint graphs, then re-label the vertices in \( G_1 \cap G_2 \) in such a way that the difference sets of the set-labels of adjacent vertices in \( G_1 \cap G_2 \) and the last vertices of \( G_1 \cap G_2 \) and their adjacent vertices in \( G_1 \) and \( G_2 \) follow the relation ‘\(<\)’. Hence, \( f \) is a strong IASI of \( G_1 \cup G_2 \).

Conversely, assume that \( G_1 \cup G_2 \) admits a strong IASI. Hence, by Theorem [1.7] being the subgraphs of a strong IASI-graph \( G_1 \cup G_2 \), \( G_1 \) and \( G_2 \) admit the (induced) strong IASIs, \( f|_{G_1} \) and \( f|_{G_2} \), where \( f|_{G_1} \) is the restriction of \( f \) to the (sub)graph \( G_i \). \[ \square \]
Theorem 2.8. Let $G_1$ and $G_2$ be two strong IASI-graphs. Then, $\kappa(G_1 \cup G_2) \geq \max(\kappa(G_1), \kappa(G_2))$.

Proof. Let $G_1$ and $G_2$ be two strong IASI-graphs. Let $H_1$ and $H_2$ be the maximal cliques in $G_1$ and $G_2$ respectively. If $G_1$ and $G_2$ are disjoint, so are $H_1$ and $H_2$. Without loss of generality, let $|V(H_1)| \geq |V(H_2)|$. Then, $H_1$ is the maximal clique in $G_1 \cup G_2$. Hence, $\kappa(G_1 \cup G_2) = \kappa(G_1)$. Therefore, in general, for disjoint graphs $G_1$ and $G_2$, $\kappa(G_1 \cup G_2) = \max(\kappa(G_1), \kappa(G_2))$. If $G_1$ and $G_2$ are not disjoint, then there may exist a subgraph $H'_1$, not necessarily complete, in $G_1$ and a subgraph $H'_2$, not necessarily complete, in $G_2$ such that $H'_1 \cup H'_2 = K_l$, where $l \geq |V(H_1)|, |V(H_2)|$. In this case, $\kappa(G_1 \cup G_2) \geq \max(\kappa(G_1), \kappa(G_2))$. This completes the proof. \qed

Invoking Theorem 2.8, we observe the following theorem.

Theorem 2.9. Let $G_1$ and $G_2$ be two strong IASI-graphs. Then, $\kappa(G_1 \cup G_2) = \max(\kappa(G_1), \kappa(G_2))$ if $G_1 \cap G_2$ is triangle-free.

Proof. Every complete graph $K_n$ with more than two vertices contains triangles. Hence, since $G_1 \cap G_2$ is triangle-free, it does not contain any clique. Therefore, there does not exist a subgraph $H'$ in $G_i$, $i = 1, 2$ such that $H'_1 \cup H'_2 = K_l$, $l > 2$. Hence, $\kappa(G_1 \cup G_2) = \max(\kappa(G_1), \kappa(G_2))$. \qed

The above theorem may not hold for the union of two graphs whose intersection contains triangles. For example, consider two cycles $C_m$ and $C_n$ such that $C_m \cap C_n = K_2$. We know that $\kappa(C_m) = \kappa(C_n) = 2$. But, $\kappa(C_m \cup C_n) = 3$ which is not equal to $\max(\kappa(C_m), \kappa(C_n))$.

Hence, we have the following theorem for the union of two graphs.

Theorem 2.10. Let $G_1$ and $G_2$ be two strong IASI graphs. Then, the nourishing number of $G$ is

$$
\kappa(G_1 \cup G_2) = \begin{cases} 
\max(3, \kappa(G_1), \kappa(G_2)) & \text{if } K_3 \subseteq G_1 \cap G_2 \\
\max(\kappa(G_1), \kappa(G_2)) & \text{if } K_3 \nsubseteq G_1 \cap G_2.
\end{cases}
$$

The following theorem is a necessary and sufficient condition for the admissibility of strong IASI by the join of two strong IASI-graphs.

Theorem 2.11. Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two strong IASI-graphs. Then, their join $G_1 + G_2$ admits a strong IASI if and only if the difference set of the set-label of every vertex in $G_1$ is disjoint from the difference sets of the set-labels of all vertices of $G_2$.

Proof. Let $V(G_1) = \{u_1, u_2, u_3, \ldots, u_m\}$ and $V(G_2) = \{v_1, v_2, v_3, \ldots, v_n\}$. Now let $E_3 = \{u_i \colon u_i \in V(G_1), v_j \in V(G_2)\}$. Let $G_3$ be the subgraph of $G_1 + G_2$ with the edge set $E_3$. Therefore, $G_1 + G_2 = G_1 \cup G_2 \cup G_3$. Also, let $f_1, f_2, f_3$ be the IASIs defined on $G_1, G_2, G_3$ respectively. Given that $f_1$ and $f_2$ are strong IASIs. Let $f$ be an IASI defined on $G_1 + G_2$ by $f(v) = f_i(v)$ if $v \in V(G_i)$, $i = 1, 2, 3$. Then, by Theorem 2.7 $f$ is a strong IASI if and only if $f_3$ is a strong IASI.

First assume that $f$ is a strong IASI. Then, $f_3$ is also a strong IASI. Hence, $|g_{f_3}(u_i; v_j)| = |f_3(u_i)| \cdot |f_3(v_j)| \forall \ u_i \in V(G_1), \ v_j \in V(G_2)$. By Lemma 1.5, the difference sets of the set-labels of these vertices follow the relation $D_{u_i} < D_{v_j}, \forall \ u_i \in V(G_1), \ v_j \in V(G_2)$.

Conversely, assume that $D_{u_i} < D_{v_j}, \forall \ u_i \in V(G_1), \ v_j \in V(G_2)$. Then, $|g_{f_3}(u_i; v_j)| = |f_3(u_i)| \cdot |f_3(v_j)| \forall \ u_i, v_j \in E_3$. Therefore, $f_3$ is a strong IASI on $G_3$ and hence $f$ is a strong IASI on $G_1 + G_2$. This completes the proof. \qed
The nourishing number of the join of two strong IASI-graphs has been determined in the following theorem.

**Theorem 2.12.** Let $G_1$ and $G_2$ be two strong IASI-graphs. Then, $\kappa(G_1 + G_2) = \kappa(G_1) + \kappa(G_2)$.

**Proof.** Let $H_1$ be a maximal clique with order $m$ in $G_1$ and $H_2$ be a maximal clique with order $n$ in $G_2$. Since every vertex of $H_1$ is joined to every vertex of $G_2$ in $G_1 + G_2$, it is so in $H_1 + H_2$ also. Since $H_1$ and $H_2$ are cliques, they are complete graphs. Hence, every vertex of $H_1$ is adjacent to all other vertices of $H_1$ and all vertices of $H_2$. Similarly, every vertex of $H_2$ is adjacent to all other vertices of $H_2$ and all vertices of $H_1$. Hence, $H_1 + H_2$ is an $(m + n - 1)$-regular graph on $m + n$ vertices. Therefore, $H_1 + H_2$ is a complete graph and hence is a clique in $G_1 + G_2$. Since $H_1$ and $H_2$ are maximal cliques, $H_1 + H_2$ is maximal in $G_1 + G_2$. Hence, $\kappa(G_1 + G_2) = m + n = \kappa(G_1) + \kappa(G_2)$. \hfill \Box

Next, let us discuss the nourishing number of the complement of a strong IASI-graph $G$. A graph $G$ and its complement $\bar{G}$ have the same set of vertices and hence $G$ and $\bar{G}$ have the same set-labels for their corresponding vertices. We observe that the strong IASIs, except some, defined on $G$ do not induce strong IASI on $\bar{G}$. A set-labeling of $V(G)$ that defines a strong IASI for both the graphs $G$ and its complement $\bar{G}$ may be called a strongly concurrent set-labeling.

Theorem 2.13. Let $G$ be a strong IASI-graph on $n$ vertices and let $\bar{G}$ be its complement. Then, $\bar{G}$ admits a strong IASI if and only if the length of the chain of difference sets of set-labels of vertices in $G$ or in $\bar{G}$ is $n$.

**Proof.** We have $G \cup \bar{G} = K_n$. Therefore, the length of the chain of difference sets in $G \cup \bar{G}$ is $n$. Since $\kappa(G \cup \bar{G}) \geq \max(\kappa(G), \kappa(\bar{G}))$, the length of the chain of difference sets of set-labels of vertices in $G$ or in $\bar{G}$ must be $n$.

Conversely, assume that the length of the chain of difference sets of set-labels of vertices in $G$ or in $\bar{G}$ is $n$. Then, length of the chain of difference sets in $G \cup \bar{G}$ is $n$, which is the maximum possible length of a chain of difference sets. Therefore, both $G$ and $\bar{G}$ admit strong IASI under the same set-labels for the vertices of $G$. This completes the proof. \hfill \Box

Corollary 2.14. If $G$ is a self-complementary graph on $n$ vertices, which admits a strong IASI, then $\kappa(G) = n$.

**Proof.** Since $G$ is self complementary, we have $G \cong \bar{G}$ and hence $\kappa(G) = \kappa(\bar{G})$. Therefore, \[ n = \kappa(G \cup \bar{G}) = \max(\kappa(G), \kappa(\bar{G})) = \kappa(G). \] That is, $\kappa(G) = \kappa(\bar{G}) = n$. Therefore, $G$ and $\bar{G}$ admit strong IASI. This completes the proof. \hfill \Box

2.3 Strong IASIs of Graph Products

In this section, we discuss the admissibility of strong IASI by certain products of strong IASI graphs. In graph products, we may have to make certain number of copies of the graphs and to make suitable of attachments between these copies and the given graphs. Therefore, we have
to establish a suitable IASI, if exists, to each of these copies. Hence, we make the following remark.

**Remark 2.15.** Let $G_i$ is a copy of a given graph $G$, which appears in a graph product. Let $n \cdot A = \{na_i : a_i \in A\}$, for $n \in \mathbb{N}_0$. Note that $n \cdot A \neq nA$. If $f$ is a strong IASI on $G$, then the $i$-th copy of $G$ denoted by $G_i$ has the set-label $f_i$ where $f_i(v_i) = r \cdot f(v)$, $r \in \mathbb{N}$, where $v_i$ is the vertex in $G_i$ corresponding to the vertex $v$ in $G$. We observe that if two sets $A$ and $B$ are disjoint, then $n \cdot A$ and $n \cdot B$ are also disjoint. Hence, if $f$ is a strong IASI of $G$, then $f_i$ is a strong IASI of $G_i$.

In this section, we verify the admissibility of strong IASI by the Cartesian product of two graphs. In this section, by the term product of graphs we mean the Cartesian product of graphs. we recall the definition the Cartesian product of two graphs as follows.

Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two graphs. Then, the Cartesian product or simply product of $G_1$ and $G_2$, denoted by $G_1 \square G_2$, is the graph with vertex set $V_1 \times V_2$ defined as follows. Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be two points in $V_1 \times V_2$. Then, $u$ and $v$ are adjacent in $G_1 \square G_2$ whenever $[u_1 = v_1$ and $u_2$ is adjacent to $v_2]$ or $[u_2 = v_2$ and $u_1$ is adjacent to $v_1]$. If $|V_1| = p_i$ and $|E_i| = q_i$ for $i = 1, 2$, then $|V(G_1 \square G_2)| = p_1 p_2$ and $|E(G_1 \square G_2)| = p_1 q_2 + p_2 q_1$.

**Remark 2.16.** We observe that the product $G_1 \square G_2$ is obtained as follows. Make $p_2$ copies of $G_1$. Denote these copies by $G_{1i}$, $1 \leq i \leq p_2$, which corresponds to the vertex $v_i$ of $G_2$. Now, join the corresponding vertices of two copies $G_{1i}$ and $G_{1j}$ if the corresponding vertices $v_i$ and $v_j$ are adjacent in $G_2$. Thus, we view the product $G_1 \square G_2$ as a union of $p_2$ copies of $G_1$ and a finite number of edges connecting two copies $G_{1i}$ and $G_{1j}$ of $G_1$ according to the adjacency of the corresponding vertices $v_i$ and $v_j$ in $G_2$, where $1 \leq i, j \leq p_2$, $i \neq j$.

Hence, we make the following inferences on the admissibility of strong IASI by the product of two strong IASI-graphs.

**Theorem 2.17.** Let $G_1$ and $G_2$ be two strong IASI-graphs. Then, the product $G_1 \square G_2$ admits a strong IASI if and only if the difference sets of the set-labels of corresponding vertices different copies of $G_1$ which are adjacent in $G_1 \square G_2$ are disjoint.

**Proof.** Assume that the graphs $G_1$ and $G_2$ admit strong ISIs, say $f$ and $g$ respectively and $G_1 \square G_2$ admits a strong IASI, say $F$. Since $f$ is a strong IASI on $G$, then by Remark 2.15, the function $f_i$ defined on the $i$-th copy $G_{1i}$ of $G_1$, by $f_i(v_i) = r \cdot f(v)$, for some positive integer $r$, is a strong IASI on $G_{1i}$. If the corresponding vertices of two copies $G_{1i}$ and $G_{1j}$ are adjacent in $G_1 \square G_2$, then $|g_f(v_i v_j)| = |f(v_i)| \cdot |f(v_j)| = |f_i(v_i)| \cdot |f_j(v_j)|$, $\forall v_i \in G_{1i}$, $v_j \in G_{1j}$. Hence, $D_{v_i} < D_{v_j}$. That is, the difference sets of $f_i(v_i)$ and $f_j(v_j)$ are disjoint.

Conversely, assume that the difference sets of the set-labels of corresponding vertices different copies of $G_1$ which are adjacent in $G_1 \square G_2$ are disjoint. Also, the function $f_i$ defined on the $i$-th copy $G_{1i}$ of $G_1$, by $f_i(v_i) = r \cdot f(v)$, for some positive integer $r$, is a strong IASI on $G_{1i}$, $1 \leq i \leq |V(G_2)|$. Hence, the difference sets of the set-labels of all the adjacent vertices in $G_1 \square G_2$ are disjoint. Therefore, $G_1 \square G_2$ admits a strong IASI.

Invoking the above theorem, we determine the nourishing number of the Cartesian product of two strong IASI-graphs in the following theorem.
Theorem 2.18. Let $G_1$ and $G_2$ be two graphs which admit strong IASIs. Then, $\kappa(G_1 \boxdot G_2) = \max\{\kappa(G_1), \kappa(G_2)\}$.

Proof. Let $H_1$ and $H_2$ be the maximal clique in $G_1$ and $G_2$ respectively. Without loss of generality, let $H_1$ be greater than $H_2$ in terms of the number of vertices in them. For $1 \leq i \leq |V(G_2)|$, let $H_{1_i}$ be the copy of $H_1$ in $G_1$, and is the maximal clique in $G_1$. Now, observe that no vertex of another copy $G_j$ is adjacent to all the vertices of $H_{1_i}$. Since, all cliques $H_{1_i}$ are isomorphic, for $1 \leq i \leq |V(G_2)|$, $H_{1_i}$ is the maximal clique in $G_1 \boxdot G_2$. Hence, $\kappa(G_1 \boxdot G_2) = |V(H_{1_i})| = V(H_1)|$. Therefore, in general, $\kappa(G_1 \boxdot G_2) = \max\{\kappa(G_1), \kappa(G_2)\}$.

Another graph product we consider in this occasion is the corona of two graphs. The corona of two graphs $G_1$ and $G_2$, denoted by $G_1 \odot G_2$, is the graph obtained by taking $|V(G_1)|$ copies of $G_2$ and then joining the $i$-th vertex of $G_1$ to every vertex of the $i$-th copy of $G_2$.

The following result establishes the admissibility of a strong IASI by the corona of two IASI-graphs.

Theorem 2.19. Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two strong IASI-graphs. Then, their corona $G_1 \odot G_2$ admits a strong IASI if and only if the difference set of the set-label of every vertex in $G_1$ is disjoint from the difference sets of the set-labels of all vertices of the corresponding copy of $G_2$.

Proof. Let $f$ and $g$ be the strong IASI on $G_1$ and $G_2$ respectively. Let $g_i = r \cdot g$, $1 \leq i \leq |V(G_1)|$, $r$ being a positive integer, be an IASI defined on the $i$-th copy $G_i$ of $G$. Then, by Remark 2.15, $g_i$ is a strong IASI on $G_i$ for all $i = 1, 2, 3, \ldots, |V(G_1)|$.

First assume that $G_1 \odot G_2$ admits a strong IASI. Also, each copy of $G_2$ admits strong IASIs. Since $G_1 \odot G_2$ admits a strong IASI, the difference set of the set-label of each vertex $u_i$ of $G_1$ and the difference set of the set-label of each vertex $v_j$, where $1 \leq i \leq |V(G_1)|$, $1 \leq j \leq |V(G_2)|$, must be disjoint.

Conversely, assume that the difference set of the set-label of every vertex in $G_1$ is disjoint from the difference sets of the set-labels of all vertices of the corresponding copy of $G_2$. Since each copy of $G_2$ is also strong IASI-graph, for every pair of adjacent vertices $u, v$ in $G_1 \odot G_2$, the difference sets $D_u$ and $D_v$ hold the relation $D_u < D_v$. Hence, $G_1 \odot G_2$ admits a strong IASI. □

In view of the above theorem, the nourishing number of the corona of two strong IASI-graphs, is determined in the theorem given below.

Theorem 2.20. If $G_1$ and $G_2$ are two strong IASI-graphs, then

$$
\kappa(G_1 \odot G_2) = \begin{cases} 
\kappa(G_1) & \text{if } \kappa(G_1) > \kappa(G_2) \\
\kappa(G_2) + 1 & \text{if } \kappa(G_2) > \kappa(G_1). 
\end{cases}
$$

Proof. Let $H_1$ and $H_2$ be the maximal cliques in $G_1$ and $G_2$ respectively. Then, $H_2$ is the copy of $H_2$ in $G_2$, which is maximal in $G_2$. Since the vertex $u_i$ of $H_1$ is adjacent to all vertices of the copy $H_2$ in $G_1 \odot G_2$, we can find $|V(G_1)|$ cliques in $G_1 \odot G_2$ with clique number $1 + |V(H_{2_i})| = 1 + |V(H_2)| = 1 + \kappa(G_2)$.

If $\kappa(G_1) > \kappa(G_2)$, then clearly, $\kappa(G_1 \odot G_2) = \kappa(G_1)$. If $\kappa(G_1) < \kappa(G_2)$, then the maximal clique in $G_1 \odot G_2$ is $H_{2_i} + \{u_i\}$. Therefore, $\kappa(G_1 \odot G_2) = 1 + \kappa(G_2)$. This completes the proof. □
3. Conclusion

In this paper, we have reviewed the properties and characteristics of certain strong IASI-graphs and studied the admissibility of strong IASIs by certain graph classes, graph operations and graph products and determined the nourishing numbers. The admissibility of strong IASI by various other graph classes, operations and products are yet to be verified and finding their corresponding nourishing numbers are to be estimated.

More properties and characteristics of strong IASIs, both uniform and non-uniform, are yet to be investigated. The problems of establishing the necessary and sufficient conditions for various graphs and graph classes to have certain IASIs still remain unsettled. All these facts highlight a great scope for further studies in this area.

Competing Interests

The authors declare that they have no competing interests.

Authors’ Contributions

Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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A Characterisation of Strong Integer Additive Set-Indexers of Graphs: N.K. Sudev and K.A. Germina


