Communications in Mathematics and Applications
Volume 3 (2012), Number 3, pp. 333–344

http://www.rgnpublications.com

On WH Packets in $L^2(\mathbb{R})$

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Abstract. WH packets with respect to a Gabor system (frame) have been introduced and it has been shown with the help of examples that WH packets with respect to a Gabor frame may not be a frame for $L^2(\mathbb{R})$. A sufficient condition under which WH packets with respect to a Gabor frame is a frame for $L^2(\mathbb{R})$ has been given. A necessary and sufficient condition has also been given in this direction. Further, finite sum of Gabor frames has been considered and sufficient conditions under which finite sum of Gabor frames is a frame for $L^2(\mathbb{R})$ have been given. Finally, stability of Gabor frames has been studied and sufficient conditions in this direction have been obtained.

1. Introduction

Fourier transform has been a major tool in analysis for over a century. It has a serious lacking for signal analysis in which it hides in its phases information concerning the moment and duration of a signal. What was needed was a localized time frequency representation which has this information encoded in it. In 1946, Dennis Gabor [8] filled this gap and formulated a fundamental approach to signal decomposition in terms of elementary signals. On the basis of this development, in 1952, Duffin and Schaeffer [6] introduced frames for Hilbert spaces to study some deep problems in non-harmonic Fourier series. In fact, they abstracted the fundamental notion of Gabor for studying signal processing. The idea of Duffin and Schaeffer did not generate much interest outside non-harmonic Fourier series. It took more than 30 years to realize the importance and potential of frames.

In 1986, Daubechies, Grossmann and Meyer [5] reintroduced frames and observed that frames can be used to find series expansions of function in $L^2(\mathbb{R})$, which are similar to the expansion using orthonormal basis. At this point, researchers started realizing the importance and potential of frames. For an introduction to frames, one may refer to [1, 2, 9].

2010 *Mathematics Subject Classification.* 42C15, 42C30, 46B15.
Key words and phrases. Frames; Gabor frames; WH packets.
The research of third author is supported by the CSIR (India) (vide letter No. 09/172(0053)/2006-EMR-I dated 19.07.06).

Frames are main tools for use in signal and image processing, data compression, sampling theory, optics, filter banks, signal detection, time frequency analysis as well as in the study of Besov spaces. Besides traditional applications, frames are now used to mitigate the effect of losses in packet-based communication systems and hence to improve the robustness of data transmission and to design high-rate constellations with full diversity in multiple-antenna code design. Frames are also used in wireless sensor networks, geophones in geophysics measurements and studies, and in the physiological structure of visual and hearing systems.

There is a long tradition for studying the stability of various notions under perturbation. Favier and Zalik [7], and Christensen [3, 4] studied stability of frames under perturbation and proved various results including a result for frames, with a perturbation condition that generalizes the Paley-Wiener condition used in the context of bases.

In the present paper, we introduced WH packets with respect to a Gabor system and exhibited with the help of examples that WH packets with respect to a Gabor frame may not be a frame for $L^2(\mathbb{R})$. A sufficient condition in this regard has been obtained. Also, a necessary and sufficient condition for WH packets with respect to a Gabor frame to be a frame for $L^2(\mathbb{R})$ has been obtained. Further, finite sum of Gabor frames has been considered and sufficient condition under which finite sum of Gabor frames is a frame for $L^2(\mathbb{R})$ have been given. Finally, stability of Gabor frames has been studied and sufficient conditions in this regard have been obtained.

2. **Preliminaries**

Throughout the paper, H will denote an infinite dimensional Hilbert space. The cardinality of a set D is denoted by $|D|$.

Given a function $f \in L^2(\mathbb{R})$, we define

$$T_a f(x) : f(x-a), \text{ for } a \in \mathbb{R}; \qquad \text{(Translation operator)}$$

$$E_a f(x) : e^{2\pi iax} f(x), \text{ for } a \in \mathbb{R}. \qquad \text{(Modulation operator)}$$

Each of these is a unitary operator from $L^2(\mathbb{R})$ onto itself and

$$E_b T_a f(x) = e^{2\pi iax} f(x-a), \quad \text{for } a, b \in \mathbb{R}.$$

For any given $g \in L^2(\mathbb{R})$, a sequence of type $\{E_{mb} T_{na} g\}_{m,n\in\mathbb{Z}}$ is called a *Gabor system* or *Weyl-Heisenberg* (*WH*) *system*.

A sequence $\{\alpha_n\}$ is said to be *positively confined* if $0 < \inf\limits_{1\le n<\infty} \alpha_n \le \sup\limits_{1\le n<\infty} < \infty$. We denote the closed linear span of $\{f_n\}$ by $[f_n]$. A sequence $\{f_n\} \subset H$ is *complete* in H if $[f_n] = H$.

Definition 2.1. A sequence $\{f_n\}$ in a Hilbert space H is called a *frame* for H if there exist constants A and B with $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_n |\langle f, f_n\rangle|^2 \le B\|f\|^2, \quad \text{for all } f \in H. \tag{2.1}$$

The positive constants A and B, respectively, are called the *lower* and the *upper frame bounds* of the frame $\{f_n\}$. The inequality (2.1) is called the *frame inequality* for the frame $\{f_n\}$.

The frame $\{f_n\}$ is called *tight* if it is possible to choose A and B satisfying inequality (2.1) with $A = B$ and is called *normalized tight* if $A = B = 1$. The frame $\{f_n\}$ is called *exact* if removal of any arbitrary f_n renders the collection $\{f_n\}$ no longer a frame for H. A sequence $\{f_n\} \subset H$ is called a *Bessel sequences* if it satisfies upper frame inequality in (2.1). A sequence $\{f_n\}$ is said to be a *frame sequence* for H if $\{f_n\}$ is a frame for $[f_n]$.

Definition 2.2. Let $g \in L^2(\mathbb{R})$ and $a, b > 0; a, b \in \mathbb{R}$ be given. Then, the Gabor system $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is called a *Gabor frame* if it is a frame for $L^2(\mathbb{R})$, i.e., if there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}g, f\rangle|^2 \leq B\|f\|^2, \quad \text{for all } f \in L^2(\mathbb{R}).$$

The real numbers a and b, respectively, are called *translation* and *modulation parameters*. The function g is called a *window function* and A, B are called the *frame bounds* for the frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$.

3. **WH Packets**

In this section, we introduce and study WH packets with respect to a Gabor system $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$.

Definition 3.1. Let, for some $g \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}$, $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ be a given WH system or Gabor system. We call $\{g_{i,j}\}_{i,j\in\mathbb{Z}}$ *WH packets* with respect to $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ if it is of the form

$$g_{i,j} = \sum_{(m,n)\in D_{i,j}} \alpha_{m,n} E_{mb}T_{na}g, \quad (i, j \in \mathbb{Z}),$$

where $D_{i,j}$'s are such that $\bigcup\limits_{i,j\in\mathbb{Z}} D_{i,j} = \mathbb{Z}\times\mathbb{Z}$, $D_{i',j'} \cap D_{i,j} = \phi$, $i, j, i', j' \in \mathbb{Z}$, $i \neq i'$ or $j \neq j'$ and $\alpha_{m,n}$'s are any scalars.

In view of the definition of WH packets, we have the following observations.

Observation 3.2. *WH packets with respect to a Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$ may be a frame for $L^2(\mathbb{R})$. Indeed, if we choose $\alpha_{m,n} = 1$, $m, n \in \mathbb{Z}$ and $D_{i,j} = \{(i, j)\}$, $i, j \in \mathbb{Z}$, then $\{g_{i,j}\}_{i,j\in\mathbb{Z}} = \{E_{ib}T_{ja}g\}_{i,j\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$.*

Observation 3.3. *WH packets may fail to be a frame for $L^2(\mathbb{R})$. Indeed, let $\{E_mT_n\chi_{[0,1[}\}_{m,n\in\mathbb{Z}}$ be a Gabor frame for $L^2(\mathbb{R})$ and let $\alpha_{m,n} = 1$, $m, n \in \mathbb{Z}$. Define*

$D_{i,j}$'s by

$$\begin{cases} D_{1,1} = \{(i,j)\}, & 1 \le i \le 3,\ 1 \le j \le 3,\ i,j \in \mathbb{Z} \\ D_{i,j} = \{(i+2, j+2)\}, & i \ge 2,\ j \ge 2,\ i,j \in \mathbb{Z} \\ D_{i,j} = \{(i,j)\}, & i \le 0,\ j \le 0,\ i,j \in \mathbb{Z}. \end{cases}$$

Then, $\{g_{i,j}\}_{i,j\in\mathbb{Z}}$ is not complete in $L^2(\mathbb{R})$ and hence is not a frame for $L^2(\mathbb{R})$.

In view of Observations 3.2 and 3.3, it is natural to ask for conditions under which WH packets are also frames for $L^2(\mathbb{R})$. In this direction, we prove the following sufficient condition.

Theorem 3.4. *Let $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ be any Gabor frame for $L^2(\mathbb{R})$ and $\{g_{i,j}\}_{i,j\in\mathbb{Z}}$ be WH packets with respect to $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ given by*

$$g_{i,j} = \sum_{(m,n)\in D_{i,j}} \alpha_{m,n} E_{mb} T_{na} g, \quad (i,j \in \mathbb{Z}),$$

where $D_{i,j}$'s are such that $\bigcup_{i,j\in\mathbb{Z}} D_{i,j} = \mathbb{Z}\times\mathbb{Z}$, $D_{i',j'} \cap D_{i,j} = \phi$, $i,j,i',j' \in \mathbb{Z}$, $i \ne i'$ or $j \ne j'$, $\sup_{i,j\in\mathbb{Z}} |D_{i,j}| < \infty$ and $\{\alpha_{m,n}\}$ is a positively confined sequence.
Then, $\{g_{i,j}\}_{i,j\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ if

$$\inf_{i,j\in\mathbb{Z}} \left[\sum_{(m,n)\in D_{i,j}} |\alpha_{m,n}|^2 - \sum_{\substack{(m,n),(m',n')\in D_{i,j} \\ (m,n)\ne(m',n')}} |\alpha_{m,n}||\overline{\alpha_{m',n'}}| \right] > 0$$

Proof. Let $K = \sup_{i,j\in\mathbb{Z}} |D_{i,j}| < \infty$ and A, B be bounds of Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$. Let $f \in L^2(\mathbb{R})$, then

$$\begin{aligned} \sum_{i,j\in\mathbb{Z}} |\langle f, g_{i,j}\rangle|^2 &= \sum_{i,j\in\mathbb{Z}} \left| \left\langle f, \sum_{(m,n)\in D_{i,j}} \alpha_{m,n} E_{mb} T_{na} g \right\rangle \right|^2 \\ &\le K \sum_{i,j\in\mathbb{Z}} \sum_{(m,n)\in D_{i,j}} |\alpha_{m,n}|^2 |\langle f, E_{mb}T_{na}g\rangle|^2 \\ &\le K \Big(\sup_{m,n\in\mathbb{Z}} |\alpha_{m,n}|^2 \Big) B \|f\|^2, \quad f \in L^2(\mathbb{R}). \end{aligned}$$

Also, for any $f \in L^2(\mathbb{R})$, we have

$$\begin{aligned} \sum_{i,j\in\mathbb{Z}} |\langle f, g_{i,j}\rangle|^2 &= \sum_{i,j\in\mathbb{Z}} \left| \sum_{(m,n)\in D_{i,j}} \bar{\alpha}_{m,n} \langle f, E_{mb}T_{na}g\rangle \right|^2 \\ &= \sum_{i,j\in\mathbb{Z}} \Big[\sum_{(m,n)\in D_{i,j}} |\alpha_{m,n}|^2 |\langle f, E_{mb}T_{na}g\rangle|^2 \end{aligned}$$

$$+ \sum_{\substack{(m,n),(m',n')\in D_{i,j}\\(m,n)\neq(m',n')}} \alpha_{m,n}\bar{\alpha}_{m',n'}\langle f, E_{mb}T_{na}g\rangle\langle f, E_{m'b}T_{n'a}g\rangle\Bigg]$$

$$\geq \sum_{i,j\in\mathbb{Z}}\Bigg[\sum_{(m,n)\in D_{i,j}} |\alpha_{m,n}|^2|\langle f, E_{mb}T_{na}g\rangle|^2$$

$$- \sum_{\substack{(m,n),(m',n')\in D_{i,j}\\(m,n)\neq(m',n')}} |\alpha_{m,n}||\overline{\alpha_{m',n'}}||\langle f, E_{mb}T_{na}g\rangle||\langle f, E_{m'b}T_{n'a}g\rangle|\Bigg]$$

$$\geq A\Bigg(\inf_{i,j\in\mathbb{Z}}\Bigg[\sum_{(m,n)\in D_{i,j}} |\alpha_{m,n}|^2 - \sum_{\substack{(m,n),(m',n')\in D_{i,j}\\(m,n)\neq(m',n')}} |\alpha_{m,n}||\overline{\alpha_{m',n'}}|\Bigg]\Bigg)\|f\|^2, f\in L^2(\mathbb{R}).$$

Hence $\{g_{i,j}\}_{i,j\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. □

Next, we prove a necessary and sufficient condition under which WH packets with respect to a Gabor frame is a frame for $L^2(\mathbb{R})$.

Theorem 3.5. *Let $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ be a Gabor frame for $L^2(\mathbb{R})$ and $\{g_{i,j}\}$ be WH packets with respect to $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$. Let $T:\ell_2(\mathbb{Z}^2)\to\ell_2(\mathbb{Z}^2)$ be a bounded linear operator such that*

$$T(\{\langle E_{mb}T_{an}g, f\rangle\}_{m,n\in\mathbb{Z}}) = \{\langle g_{i,j}, f\rangle\}_{i,j\in\mathbb{Z}},\quad f\in L^2(\mathbb{R}).$$

Then, $\{g_{i,j}\}_{i,j\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ if and only if there exists a constant $\lambda>0$ such that

$$\sum_{i,j\in\mathbb{Z}}|\langle g_{i,j}, f\rangle|^2 \geq \lambda\sum_{m,n\in\mathbb{Z}}|\langle E_{mb}T_{na}g, f\rangle|^2,\quad f\in L^2(\mathbb{R}).$$

Proof. Let $0<A\leq B<\infty$ be such that

$$A\|f\|^2 \leq \sum_{m,n\in\mathbb{Z}}|\langle E_{mb}T_{na}g, f\rangle|^2 \leq B\|f\|^2,\quad f\in L^2(\mathbb{R}).$$

Then, for any $f\in L^2(\mathbb{R})$, we have

$$\sum_{i,j\in\mathbb{Z}}|\langle g_{i,j}, f\rangle|^2 \geq \lambda A\|f\|^2,\quad f\in H.$$

Also

$$\sum_{i,j\in\mathbb{Z}}|\langle g_{i,j}, f\rangle|^2 = \|T(\{\langle E_{mb}T_{na}g, f\rangle\}_{m,n\in\mathbb{Z}})\|^2$$

$$\leq \|T\|^2B\|f\|^2,\quad f\in L^2(\mathbb{R}).$$

Hence $\{g_{i,j}\}$ is a frame for H.

Conversely, let $\{g_{i,j}\}_{i,j\in\mathbb{Z}}$ be a frame for $L^2(\mathbb{R})$ with frame bounds A' and B'. Then, for any $f \in L^2(\mathbb{R})$

$$\sum_{i,j\in\mathbb{Z}} |\langle g_{i,j}, f\rangle|^2 \geq A'\|f\|^2 .$$

Also, for any $f \in L^2(\mathbb{R})$

$$\frac{1}{B}\sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}g, f\rangle|^2 \leq \|f\|^2 .$$

This gives

$$\sum_{i,j\in\mathbb{Z}} |\langle g_{i,j}, f\rangle|^2 \geq \lambda \sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}g, f\rangle|^2, \quad f \in L^2(\mathbb{R}),$$

where $\lambda = \dfrac{A'}{B} > 0$.

4. **Finite Sum of Gabor Frames**

In the definition of WH packets, if we relax the condition that $D_{i,j} \cap D_{i',j'} = \phi$, $i \neq i'$ or $j \neq j'$, then we get a more generalized concept of WH packets.

In particular, if we take $g_i = g$, $1 \leq i \leq k$ and $|D_{i,j}| = k$ and $\alpha_{i,j} = 1$ for all $i, j \in \mathbb{Z}$, then we get a particular case of the generalized concept of WH packets. Thus we consider the finite sum of Gabor frames and obtained related results which will give results for WH packets on imposing particular conditions.

In this section, we shall consider the finite sum of Gabor frames and obtain conditions under which finite sum of Gabor frames for $L^2(\mathbb{R})$ is also a frame for $L^2(\mathbb{R})$.

Let $\Lambda_k = \{1, 2, \ldots, k\}$ be a finite set and $\{E_{mb}T_{na}g_i\}_{m,n\in\mathbb{Z}}$, $i \in \Lambda_k$, be Gabor frames for $L^2(\mathbb{R})$. Consider $\left\{\sum_{i=1}^{k} E_{mb}T_{na}g_i\right\}_{m,n\in\mathbb{Z}}$.

Then, $\left\{\sum_{i=1}^{k} E_{mb}T_{na}g_i\right\}_{m,n\in\mathbb{Z}}$ may not be a frame for $L^2(\mathbb{R})$.

Example 4.1. Let $g = \chi_{[0,1[}$, $h = e^{-x^2/4}$ and $a = b = 1$ be parameters. Then

(i) If $g_i = g$, for all $i \in \Lambda_k$, then $\left\{\sum_{i=1}^{k} E_{mb}T_{na}g_i\right\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$.

(ii) If $g_i = g$, for $1 \leq i \leq k-1$ and $g_k = h$, then $\left\{\sum_{i=1}^{k} E_{mb}T_{na}g_i\right\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. Here note that $\{E_{mb}T_{na}g_k\}_{m,n\in\mathbb{Z}}$ is not a frame for H.

(iii) If $g_i = g$, for $i = 1, 3, \ldots, k-1$ and $g_i = -g$, for $i = 2, 4, \ldots, k$, where k is even, then $\left\{\sum_{i=1}^{k} E_{mb}T_{na}g_i\right\}_{m,n\in\mathbb{Z}}$ is not a frame for $L^2(\mathbb{R})$. However, for each $i \in \Lambda_k$, $\{E_{mb}T_{na}g_i\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$.

Now, we give a necessary and sufficient condition under which finite sum of Gabor frames is a frame for $L^2(\mathbb{R})$.

Theorem 4.2. *Let $\{E_{mb}T_{na}g_i\}_{m,n\in\mathbb{Z}}$, $i \in \{1,2,\dots,k\}$ be Gabor frames for $L^2(\mathbb{R})$. Let $\{\alpha_i\}_{i=1}^k$ be any scalars. Then, $\left\{\sum\limits_{i=1}^k \alpha_i E_{mb}T_{na}g_i\right\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ if and only if there exists $\beta > 0$ and some $p \in \{1,2,\dots,k\}$ such that*

$$\beta \sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}g_p, f\rangle|^2 \le \sum_{m,n\in\mathbb{Z}} \left|\left\langle \sum_{i=1}^k \alpha_i E_{mb}T_{na}g_i, f\right\rangle\right|^2, \quad f \in L^2(\mathbb{R}).$$

Proof. For each $1 \le p \le k$, let A_p and B_p be the bounds of the frame $\{E_{mb}T_{na}g_p\}_{m,n\in\mathbb{Z}}$. Then, for any $f \in L^2(\mathbb{R})$

$$\beta A_p\|f\|^2 \le \sum_{m,n\in\mathbb{Z}} \left|\left\langle \sum_{i=1}^k \alpha_i E_{mb}T_{na}g_i, f\right\rangle\right|^2, \quad f \in L^2(\mathbb{R})$$

and

$$\begin{aligned}\sum_{m,n\in\mathbb{Z}} \left|\left\langle \sum_{i=1}^k \alpha_i E_{mb}T_{na}g_i, f\right\rangle\right|^2 &\le k\sum_{i=1}^k \left(|\alpha_i|^2 \sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}g_i, f\rangle|^2\right)\\ &\le k(\max|\alpha_i|^2)\left(\sum_{i=1}^k B_i\right)\|f\|^2, \quad f \in L^2(\mathbb{R}).\end{aligned}$$

Conversely, let $\left\{\sum\limits_{i=1}^k \alpha_i E_{mb}T_{na}g_i\right\}_{m,n\in\mathbb{Z}}$ be a frame for $L^2(\mathbb{R})$ with bounds A, B and let for any $p \in \{1,2,\dots,k\}$, $\{E_{mb}T_{na}g_p\}_{m,n\in\mathbb{Z}}$ be a frame for $L^2(\mathbb{R})$ with bounds A_p and B_p. Then, for any $f \in L^2(\mathbb{R})$,

$$\frac{1}{B_p}\sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}g_p, f\rangle|^2 \le \|f\|^2.$$

Also

$$A\|f\|^2 \le \sum_{m,n\in\mathbb{Z}} \left|\left\langle \sum_{i=1}^k \alpha_i E_{mb}T_{na}g_i, f\right\rangle\right|^2 \le B\|f\|^2, \quad f \in L^2(\mathbb{R}).$$

Hence

$$\beta \sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}g_p, f\rangle|^2 \le \sum_{m,n\in\mathbb{Z}} \left|\left\langle \sum_{i=1}^k E_{mb}T_{na}g_i, f\right\rangle\right|^2, \quad f \in L^2(\mathbb{R}),$$

where $\beta = A/B_p$. □

Finally in this section, we prove a sufficient condition under which finite sum of Gabor frames is a frame for $L^2(\mathbb{R})$ in terms of their bounds.

Theorem 4.3. *Let $\{E_{mb}T_{na}g_i\}_{m,n\in\mathbb{Z}}$, $i \in \{1,2,\dots,k\}$ be Gabor frames for $L^2(\mathbb{R})$ with bounds A_i and B_i respectively. Let $\{\alpha_i\}_{i=1}^{k}$ be any positive scalars. If for some $p \in \{1,2,\dots,k\}$, we have*

$$\alpha_p A_p > \sum_{\substack{i=1\\ i\neq p}}^{k} \alpha_i^2 B_i + 2 \sum_{\substack{i,j=1\\ i\neq p, j\neq p, i\neq j}}^{k} \alpha_i \bar{\alpha}_j \sqrt{B_i B_j},$$

then, $\left\{ \sum_{i=1}^{k} \alpha_i E_{mb}T_{na}g_i \right\}_{m,n\in\mathbb{Z}}$ *is a frame for $L^2(\mathbb{R})$.*

Proof. Let $f \in L^2(\mathbb{R})$ be any function. Then

$$\begin{aligned}
&\sum_{m,n\in\mathbb{Z}} \left| \left\langle \sum_{i=1}^{k} \alpha_i E_{mb}T_{na}g_i, f \right\rangle \right|^2 \\
&\quad \leq k \sum_{m,n\in\mathbb{Z}} |\alpha_1 \langle E_{mb}T_{na}g_1, f\rangle|^2 + k \sum_{m,n\in\mathbb{Z}} |\alpha_2 \langle E_{mb}T_{na}g_2, f\rangle|^2 + \dots \\
&\qquad + k \sum_{m,n\in\mathbb{Z}} |\alpha_k \langle E_{mb}T_{na}g_k, f\rangle|^2 \\
&\quad \leq k|\alpha_1|^2 B_1 \|f\|^2 + k|\alpha_2|^2 B_2 \|f\|^2 + \dots + k|\alpha_k|^2 B_k \|f\|^2 \\
&\quad = k\left(\sum_{i=1}^{k} |\alpha_i|^2 B_i \right) \|f\|^2, \quad f \in L^2(\mathbb{R}).
\end{aligned}$$

Let, for some $p \in \Lambda_k$, the condition that

$$\alpha_p A_p > \sum_{\substack{i=1\\ i\neq p}}^{k} \alpha_i^2 B_i + 2 \sum_{\substack{i,j=1\\ i\neq j, i\neq p, j\neq p}}^{k} \alpha_i \bar{\alpha}_j \sqrt{B_i B_j}$$

be satisfied.

Then, for any $f \in L^2(\mathbb{R})$, we have

$$\begin{aligned}
&\sum_{m,n\in\mathbb{Z}} \left| \left\langle \sum_{i=1}^{k} \alpha_i E_{mb}T_{na}g_i - \alpha_p E_{mb}T_{na}g_p, f \right\rangle \right|^2 \\
&\quad = \sum_{m,n\in\mathbb{Z}} \Bigg[\sum_{\substack{i=1\\ i\neq p}}^{k} |\alpha_i|^2 |\langle E_{mb}T_{na}g_i, f\rangle|^2 \\
&\qquad + \sum_{\substack{i,j=1\\ i\neq j, i\neq p, j\neq p}} \alpha_i \bar{\alpha}_j \langle E_{mb}T_{na}g_i, f\rangle \langle E_{mb}T_{na}g_j, f\rangle \Bigg] \\
&\quad \leq \sum_{m,n\in\mathbb{Z}} \Bigg[\sum_{\substack{i=1\\ i\neq p}}^{k} |\alpha_i|^2 |\langle E_{mb}T_{na}g_i, f\rangle|^2
\end{aligned}$$

$$
\left. + \sum_{\substack{i,j=1 \\ i\neq j, i\neq p, j\neq p}} |\alpha_i||\bar{\alpha}_j||\langle E_{mb}T_{na}g_i, f\rangle||\langle E_{mb}T_{na}g_j, f\rangle| \right]
$$

$$
\le \left(\sum_{\substack{i=1 \\ i\neq p}}^{k} |\alpha_i|^2 B_i + \sum_{\substack{i,j=1 \\ i\neq j, i\neq p, j\neq p}}^{k} \alpha_i \bar{\alpha}_j \sqrt{B_i B_j} \right) \|f\|^2
$$

$$
< \alpha_p A_p \|f\|^2, \quad f \in L^2(\mathbb{R}).
$$

Hence $\left\{ \sum_{i=1}^{k} \alpha_i E_{mb} T_{na} g_i \right\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. □

5. **Stability of Gabor Frames**

In this section, we discuss the stability of Gabor frames and also obtain conditions for their stability.

Let $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ be a Gabor frame and let $h \in L^2(\mathbb{R})$ be given such that $\{E_{mb}T_{na}(g+h)\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence. Then, $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ may not be a frame for $L^2(\mathbb{R})$. We give the following examples in this regard.

Example 5.1. Let $g = \chi_{[0,1[}$ and $a = b = 1$. If we choose $h = g$, then one may verify that $\{E_{mb}T_{na}(g+h)\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence for $L^2(\mathbb{R})$. Also $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ is a Gabor frame for $L^2(\mathbb{R})$.

Example 5.2. Let g, a, b be as in Example 5.1. If we take $h = \chi_{[0,1/2[}$, then $\{E_{mb}T_{na}(g+h)\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence in $L^2(\mathbb{R})$, but $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ is not a Gabor frame for $L^2(\mathbb{R})$ as $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ is not complete in $L^2(\mathbb{R})$.

To supplement the above discussion, we prove the following result.

Theorem 5.3. *Let $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ be any Gabor frame with bounds A, B and frame operator S. Let $h \in L^2(\mathbb{R})$ be any function such that $\{E_{mb}T_{na}(g+h)\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence with Bessel bound M.*
If $M < \frac{A^2\|S\|^{-1}}{2}$, then $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ is a Gabor frame for $L^2(\mathbb{R})$.

Proof. For any $f \in L^2(\mathbb{R})$, we have

$$
\sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}h, f\rangle|^2 = \sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}(g+h), f\rangle - \langle E_{mb}T_{na}g, f\rangle|^2
$$

$$
\le 2(M+B)\|f\|^2, \quad f \in L^2(\mathbb{R}).
$$

Also

$$
(A^2\|S\|^{-1} - 2M)\|f\|^2 \le A\|f\|^2 - 2M\|f\|^2
$$

$$
\le 2 \sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}h, f\rangle|^2, \quad f \in L^2(\mathbb{R}).
$$

Hence

$$\frac{1}{2}(A^2\|S\|^{-1} - 2M)\|f\|^2 \le \sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}h, f\rangle|^2$$
$$\le 2(M+B)\|f\|^2, \quad f \in L^2(\mathbb{R}). \qquad \square$$

Remark 5.4. The condition that $M < \dfrac{A^2\|S\|^{-1}}{2}$ in Theorem 5.3 is not necessary as we can see in Example 5.1, where $A = B = 1$ and $M = 4$.

Next, we give a necessary and sufficient condition for a Gabor system $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ to be a Bessel sequence in $L^2(\mathbb{R})$, in terms of some Gabor frame.

Theorem 5.5. *Let $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ be any Gabor frame for $L^2(\mathbb{R})$ with bounds A, B and let $h \in L^2(\mathbb{R})$ be any function. Then, $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence in $L^2(\mathbb{R})$ if and only if there exists an $\alpha > 0$ such that*

$$\sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}(g-h), f\rangle|^2 \le \alpha \sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}g, f\rangle|^2,$$

for all $f \in L^2(\mathbb{R})$.
Further, if there exists a bounded linear operator $U : \ell_2(\mathbb{Z}^2) \to L^2(\mathbb{R})$ given by $U(\{\langle E_{mb}T_{na}g, f\rangle\}_{m,n\in\mathbb{Z}}) = f$, for all $f \in H$, such that $\frac{(\|U\|^{-1})^2}{B} > 2\alpha$, then $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ is a Gabor frame for $L^2(\mathbb{R})$.

Proof. Let B' be a Bessel bound of the Bessel sequence $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$. Then, for any $f \in L^2(\mathbb{R})$, we have

$$\sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}(g-h), f\rangle|^2$$
$$= \sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}g, f\rangle - \langle E_{mb}T_{na}h, f\rangle|^2$$
$$\le 2\sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}g, f\rangle|^2 + 2\sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}h, f\rangle|^2$$

and

$$\sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}h, f\rangle|^2 \le B'\|f\|^2.$$

So

$$\frac{1}{B'}\sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}h, f\rangle|^2 \le \|f\|^2, \quad f \in L^2(\mathbb{R}).$$

Since $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Gabor frame,

$$\|f\|^2 \le \frac{1}{A}\sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}g, f\rangle|^2, \quad f \in L^2(\mathbb{R}).$$

Hence

$$\sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}(g-h), f\rangle|^2 \le \alpha \sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}g, f\rangle|^2, \quad f \in L^2(\mathbb{R}),$$

where $\alpha = 2\left(1+\frac{B'}{A}\right)$.

Conversely, for any $f \in L^2(\mathbb{R})$, we have

$$\begin{aligned}\sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}h, f\rangle|^2 &= \sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}(h-g), f\rangle + \langle E_{mb}T_{na}g, f\rangle|^2 \\ &\le 2\alpha \sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}g, f\rangle|^2 + 2\sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}g, f\rangle|^2 \\ &\le 2(\alpha+1)B\|f\|^2, \quad f \in L^2(\mathbb{R}).\end{aligned}$$

Hence $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence with bound $2(\alpha+1)B$.

Further, by hypotheses, for any $f \in L^2(\mathbb{R})$, we have

$$\begin{aligned}&\sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}h, f\rangle|^2 \\ &\qquad \ge \frac{1}{2}\left(\sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}g, f\rangle|^2 - 2\sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}(h-g), f\rangle|^2\right) \\ &\qquad \ge \frac{1}{2}\left(\frac{(\|U\|^{-1})^2}{B} - 2\alpha\right)A\|f\|^2, \quad f \in L^2(\mathbb{R}).\end{aligned}$$

Hence

$$\begin{aligned}\frac{1}{2}\left(\frac{(\|U\|^{-1})^2}{B} - 2\alpha\right)A\|f\|^2 &\le \sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}h, f\rangle|^2 \\ &\le 2(\alpha+1)B\|f\|^2, \quad f \in L^2(\mathbb{R}). \qquad \square\end{aligned}$$

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