On Approximative Atomic Decompositions in Banach Spaces

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Abstract. Approximative atomic decomposition for Banach spaces has been defined. A characterization for approximative atomic decompositions has been obtained. Also, it has been proved that a Banach space \( E \) has an approximative atomic decomposition if and only if it possesses bounded approximation property. Further, sufficient conditions for the existence of approximative atomic decompositions in separable Banach spaces have been obtained. Finally, as an application of approximative atomic decompositions, it has been proved that if \( E \) and \( F \) are Banach spaces having bounded approximation property, then \( E \times F \) also has bounded approximative property.

1. Introduction

Frames for Hilbert spaces were introduced by Duffin and Schaeffer \([3]\), while addressing some deep problems in non-harmonic Fourier series. Feichtinger and Gröchenig \([6]\) generalized frames for Banach spaces and called them atomic decompositions.

Atomic decompositions have played a key role in the development of wavelet theory and Gabor theory. Frazier and Jawerth \([7]\) had constructed wavelet atomic decompositions for Besov spaces which they called \( \phi \)-transform. Feichtinger \([5]\) constructed Gabor atomic decompositions for the modulation spaces which are Banach spaces similar in many respects to Besov spaces, defined by smoothness and decay conditions. Walnut \([13]\) improved many results by extending them for the case of weighted \( L^2 \) spaces. Infact, he showed that Gabor atomic decompositions in \( L^2_w(\mathbb{R}) \) need not be Hilbert frames. Feichtinger and Gröchenig \([6]\) developed a general theory which applies to an extremely broad class of function spaces and group representations.

In \([8]\), Gröchenig studied atomic decompositions as a generalization of frames. In the present paper, we shall extend this study further and introduce approximative atomic decompositions for Banach spaces. A characterization for

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approximative atomic decompositions has been given. Also, it has been proved that a Banach space $E$ has an approximative atomic decomposition if and only if it possesses bounded approximation property. Further, sufficient conditions for the existence of approximative atomic decompositions in separable Banach spaces have been obtained. Finally, as an application of approximative atomic decompositions it has been proved that if $E$ and $F$ are Banach spaces having bounded approximation property, then $E \times F$ with $\ell_1$-norm also has bounded approximation property.

2. Preliminaries

Throughout the paper, $E$ will denote an infinite dimensional Banach space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), $E^*$ and $E^{**}$, respectively, the first and second conjugate spaces of $E$, $B(E)$ the Banach space of all continuous linear mappings on $E$, $L(E, E)$ the Banach space of all linear mappings on $E$, $E_d$ an associated Banach space of scalar-valued sequences, indexed by $\mathbb{N}$, $[f_n]$ the closed linear span of $\{f_n\}$ and $\overline{[f_n]}$ the closed linear span of $\{f_n\}$ in the $\sigma(E^*, E)$-topology. A sequence $\{f_n\} \subset E^*$ is said to be complete if $[f_n] = E^*$ and total if $\{x \in E : f_n(x) = 0, n \in \mathbb{N}\} = \{0\}$.

**Definition 2.1 ([6]).** Let $E$ be a Banach space and let $E_d$ be an associated Banach space of scalar-valued sequences, indexed by $\mathbb{N}$. Let $\{x_n\}$ be a sequence in $E$ and let $\{f_n\}$ be a sequence in $E^*$. Then, the pair $\{(f_n), (x_n)\}$ is called an atomic decomposition for $E$ with respect to $E_d$, if

(a) $\{f_n(x)\} \in E_d$, for all $x \in E$

(b) there exist constants $A, B$ with $0 < A \leq B < \infty$ such that

$$A\|x\| \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E,$$

for all $x \in E$

(c) $x = \sum_{n=1}^{\infty} f_n(x)x_n$, for all $x \in E$.

The positive constants $A, B$ are called atomic bounds for the atomic decomposition $\{(f_n), (x_n)\}$.

**Definition 2.2 ([12]).** A Banach space $E$ is said to have a bounded approximation property if there exists $\lambda \geq 1$ such that the identity operator $I_E : E \to E$ can be approximated, uniformly on every compact subset of $E$, by linear operators of finite rank, of norm $\leq \lambda$, that is, if their exists a constant $\lambda \geq 1$ with the property: for every compact subset $Q \subset E$ and for every $\epsilon > 0$ there exists an endomorphism $u = u_{Q, \epsilon} \in L(E, E)$ of finite rank, of norm $\|u\| \leq \lambda$, such that $\|u(x) - x\| < \epsilon$ (for $x \in Q$).

The following result which is referred in this paper is listed in the form of a lemma
Lemma 2.3 ([11, 12]). If $E$ is a Banach space and $\{f_n\} \subset E^*$ is total over $E$, then
$E$ is linearly isometric to the associated Banach space $E_d = \{f_n(x) : x \in E\}$, where
the norm is given by $\|f_n(x)\|_{E_d} = \|x\|_E$, $x \in E$.

3. Main Results

We begin with the following definition of approximative atomic decomposition

Definition 3.1. Let $E$ be a Banach space and let $E_d$ be an associated Banach space
of scalar-valued sequences, indexed by $\mathbb{N}$. Let $\{x_n\} \subset E$ and $\{h_{n,i}\}_{n \in \mathbb{N}} \subset E^*$, where $\{m_n\}$ is an increasing sequence of positive integers. Then, the pair
$\{(h_{n,i})_{n \in \mathbb{N}}, \{x_n\}\}$ is called an approximative atomic decomposition for $E$ with respect to $E_d$, if

(a) $\{h_{n,i}(x)\}_{n \in \mathbb{N}} \subset E_d$, for all $x \in E$

(b) there exist constants $A, B$ with $0 < A \leq B < \infty$ such that

\[ A\|x\|_E \leq \|\{h_{n,i}(x)\}_{n \in \mathbb{N}}\|_{E_d} \leq B\|x\|_E, \quad \text{for all } x \in E \]

(c) $x = \lim_{n \to \infty} \sum_{i=1}^{m_n} h_{n,i}(x) x_i$, for all $x \in E$.

One may observe that if $\{(f_n), \{x_n\}\}$ is an atomic decomposition for $E$ with respect to $E_d$, then, for $h_{n,i} = f_i$, $i = 1, 2, \ldots, n$; $n \in \mathbb{N}$, $\{(h_{n,i})_{n \in \mathbb{N}}, \{x_n\}\}$ is an approximative atomic decomposition for $E$ with respect to some associated Banach space $E_{d_i}$. Indeed, we have

\[ \lim_{n \to \infty} \sum_{i=1}^{n} h_{n,i}(x) x_i = \lim_{n \to \infty} \sum_{i=1}^{n} f_i(x) x_i = x. \quad (3.1) \]

Also, if $h_{n,i}(x) = 0$, for all $i = 1, 2, \ldots, n$ and $n \in \mathbb{N}$ then by (3.1), $x = 0$. Therefore,
by Lemma 2.3, there exists an associated Banach space $E_{d_0} = \{h_{n,i}(x)_{n \in \mathbb{N}} : x \in E\}$ with the norm given by $\|\{h_{n,i}(x)\}_{n \in \mathbb{N}}\|_{E_{d_0}} = \|x\|_E$, $x \in E$, such that
$\{(h_{n,i})_{n \in \mathbb{N}}, \{x_n\}\}$ is an approximative atomic decomposition for $E$ with respect to $E_{d_0}$. However, one may note that $\{(h_{n,i})_{n \in \mathbb{N}}, \{x_n\}\}$, as such, need not be an atomic decomposition for $E$ with respect to $E_{d_0}$.

In view of the above discussion, we have the following observations/queries:

(I) If $E$ has an atomic decomposition with respect to $E_d$, then $E$ also has an
approximative atomic decomposition with respect to some $E_{d_i}$.

(II) What is the relation between $E_d$ and $E_{d_i}$?

(III) Is the converse of observation (I) true?

The answer to query (II) is affirmative (Remark 3.4) and query (II) is a matter
of investigation and is open.

Regarding the existence of approximative atomic decompositions, we give the
following characterization
A Banach space E has an approximative atomic decomposition if and only if there exists a sequence \( \{v_n\} \subset B(E) \) of non-zero endomorphism of finite rank such that \( x = \sum_{i=1}^{n} v_i(x) \), \( x \in E \) and \( \sup \left\| \sum_{i=1}^{n} v_i \right\| \leq \lambda \), for some \( \lambda > 0 \).

**Proof.** Let \( \{x_n\} \subset E \) and \( \{h_{n,i}\} \subset E^* \) be the sequences such that \( \{\{h_{n,i}\}_{i=1}^{m_i} \} \) is an approximative atomic decomposition for E with respect to associated Banach space \( E_1 \), where \( \{m_i\} \) is an increasing sequence of positive integers. Define

\[
u_n(x) = \sum_{i=1}^{m_i} h_{n,i}(x) x_i, \quad x \in E, n \in \mathbb{N}.
\]

Then, for each \( n \in \mathbb{N} \), \( u_n \) is a well defined continuous linear mapping on \( E \) with \( \text{dim } u_n(E) < \infty \) such that \( \lim_{n \to \infty} u_n(x) = x \), \( x \in E \). Also, by using principle of uniform boundedness, \( \sup_{1 \leq n < \infty} \|u_n\| < \infty \). Without any loss of generality, we may assume that \( u_1 \neq 0 \) and \( u_n \neq u_{n+1} \), for all \( n \in \mathbb{N} \). Define

\[
u_1 = u_1, \quad \nu_{2n} = \nu_{2n+1} = \frac{1}{2}(u_{n+1} - u_n), \quad n \in \mathbb{N}.
\]

Then \( \{\nu_n\} \) is a sequence of non-zero endomorphisms of finite rank in \( B(E) \) such that

\[
\sum_{i=1}^{n} \nu_i(x) = u_1(x) + \frac{1}{2} \left[ (u_2(x) - u_1(x)) + (u_2(x) - u_1(x)) \right] \\
+ \frac{1}{2} \left[ (u_3(x) - u_2(x)) + (u_3(x) - u_2(x)) \right] + \cdots \\
= u_n(x), \quad x \in E.
\]

Therefore, we have

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \nu_i(x) = \lim_{n \to \infty} u_n(x) = x, \quad x \in E.
\]

Also

\[
\sup_{1 \leq n < \infty} \left\| \sum_{i=1}^{n} \nu_i \right\| = \sup_{1 \leq n < \infty} \|u_n\| < \infty.
\]

Conversely, taking \( u_n = \sum_{i=1}^{n} \nu_i, n \in \mathbb{N} \), we have \( \lim_{n \to \infty} u_n(x) = x \), \( x \in E \). Since for each \( n \in \mathbb{N} \), \( u_n(E) \) is finite dimensional, there exist a sequence \( \{g_{n,i}\}_{i=m_{n-1}+1}^{m_n} \) in \( E \) and a total sequence \( \{g_{n,i}\}_{i=m_{n-1}+1}^{m_n} \) in \( E^* \) such that

\[
u_n(x) = \sum_{i=m_{n-1}+1}^{m_n} g_{n,i}(x) y_{n,i}, \quad x \in E, \quad n \in \mathbb{N},
\]
where \( \{m_n\} \) is an increasing sequence of positive integers with \( m_0 = 0 \). Define \( \{x_n\} \subseteq E \) and \( \{h_{n,i}\}_{i=1}^{m_n} \subseteq E^* \) by

\[
x_i = y_{n,i}, \quad i = m_{n-1} + 1, \ldots, m_n, \quad n \in \mathbb{N}
\]

and

\[
h_{n,i} = \begin{cases} 0, & \text{if } i = 1, 2, \ldots, m_{n-1} \\ g_{n,i} & \text{if } i = m_{n-1} + 1, \ldots, m_n. \end{cases}
\]

Then, for each \( x \in E, n \in \mathbb{N} \),

\[
\lim_{n \to \infty} \sum_{i=1}^{m_n} h_{n,i}(x)x_i = \lim_{n \to \infty} u_n(x) = x. \tag{3.2}
\]

Let \( x \in E \) be such that \( h_{n,i}(x) = 0 \), for all \( i = 1, \ldots, m_n \) and \( n \in \mathbb{N} \), then by (3.2), \( x = 0 \). Therefore, by Lemma 2.3, there exists an associated Banach space \( E_d = \{h_{n,i}(x)\}_{i=1}^{m_n} : x \in E \} \) with norm given by \( \|h_{n,i}(x)\| = \|x\|_{E}, x \in E \).

Hence, \( \{h_{n,i}\}_{i=1}^{m_n}, \{x_n\} \) is an approximative atomic decomposition for \( E \) with respect to \( E_d \).

**Note.** For the converse part of Theorem 3.2, we do not require the assumption that

\[
\sup_{1 \leq n < \infty} \left\| \sum_{i=1}^{n} v_i \right\| < \infty.
\]

Let \( E = \ell^\infty \). For each \( n \in \mathbb{N} \), define \( v_n : E \to E \) by

\[
v_n(x) = \xi_n e_n, \quad x = \{\xi_n\}_{i=1}^{\infty} \in \ell^\infty
\]

Then \( \|v_n(x)\| = \|\xi_n e_n\| = \|\xi_n\| \leq \|x\|, n \in \mathbb{N} \). Therefore \( v_n \in B(E), n \in \mathbb{N} \). Further, note that \( \dim v_n(E) = \dim(\text{span}\{e_n\}) = 1 \) and \( \sup_{1 \leq n < \infty} \left\| \sum_{i=1}^{n} v_i \right\| = 1 \). But \( E \) is non-separable. Hence \( E \) does not have an approximative atomic decomposition.

In view of the above discussion, we have the following problem

**Problem.** Does a Banach space \( E \) has an approximative atomic decomposition (or atomic decomposition) if there exists a sequence \( \{v_n\} \subseteq B(E) \) such that \( x = \sum_{i=1}^{\infty} v_i(x), \) \( x \in E \)?

The next result gives the relation between approximative atomic decomposition and bounded approximation property.

**Theorem 3.3.** A Banach space \( E \) has approximative atomic decomposition if and only if \( E \) has bounded approximation property.

**Proof.** Let \( \{h_{n,i}\}_{i=1}^{m_n}, \{x_n\} \) be an approximative atomic decomposition for \( E \) with respect to \( E_d \), where \( \{m_n\} \) is an increasing sequence of positive integers. Therefore, proceeding as in Theorem 3.2, there exists a sequence of continuous
linear mappings \( \{u_n\} \) with \( \text{dim } u_n(E) < \infty \) such that \( \lim_{n \to \infty} u_n(x) = x, \, x \in E \) and \( \sup_{1 \leq n < \infty} ||u_n|| < \infty \). Let \( Q \) be any compact subset of \( E \). Let \( \varepsilon > 0 \) be arbitrary and \( \{y_i\}_{i=1}^r \) be a finite \( \frac{\varepsilon}{2(1+\lambda)} \)-net for \( Q \), where \( \lambda = \sup_{1 \leq n < \infty} ||u_n|| < \infty \). Then, for any \( x \in Q \), there exists some \( j \) such that
\[
||x - y_j|| < \frac{\varepsilon}{2(1+\lambda)}, \quad 1 \leq j \leq r.
\]
Since for each \( i = 1, 2, \ldots, r \), \( y_i \in E \), there exists a positive integer \( n_i \), such that
\[
||y_i - u_n(y_i)|| < \frac{\varepsilon}{2}, \quad \text{for all} \quad n \geq n_i, \quad \text{and} \quad i = 1, 2, \ldots, r.
\]
Thus
\[
||x - u_n(x)|| \leq ||x - y_j|| + ||y_j - u_n(y_j)|| + ||u_n(y_j) - u_n(x)|| < \varepsilon, \quad \text{for all} \quad n \geq n_i.
\]
Since \( x \in Q \) and \( \varepsilon > 0 \) is an arbitrary, we have
\[
\lim_{n \to \infty} \sup_{x \in Q} ||x - u_n(x)|| = 0.
\]
Hence \( E \) has bounded approximation property.

Converse part of the theorem follows from Theorem 2.10 in [1] and Observation (I).

**Remark 3.4.** In view of Theorem 3.3 and Theorem 2.10 in [1], we conclude that a Banach space \( E \) has an approximative atomic decomposition if and only if \( E \) has an atomic decomposition.

One may observe that if \( E \) has an approximative atomic decomposition, then \( E \) is separable. However, in view of Remark 3.4 and the example in [4], the converse need not be true. We give below a diagram depicting relationship among various concepts.
Since a separable Banach space need not have an approximative atomic decomposition, it is natural to ask for conditions under which a separable Banach space has an approximative atomic decomposition. The following results give sufficient conditions for the same.

**Theorem 3.5.** Let \( E \) be a separable Banach space and \( \{G_n\} \) be any sequence of subspaces of \( E \) such that for each \( n \in \mathbb{N} \), \( \dim G_n < \infty \), \( G_n \subset G_{n+1} \) and \( \bigcup_{n=1}^{\infty} G_n \) is dense in \( E \). If for each \( n \in \mathbb{N} \), there exists a finite rank endomorphism \( u_n \subset L(E,E) \) such that \( u_n|_{G_n} = I_{G_n} \) and \( \sup_{1 \leq n < \infty} \|u_n\| < \infty \), then \( E \) has an approximative atomic decomposition.

**Proof.** Let \( \{y_n\} \) be a sequence in \( E \) such that \( [y_n] = E \). Put \( G_n = [y_1, y_2, \ldots, y_n] \), \( n \in \mathbb{N} \). Then, \( \{G_n\} \) is a sequence of subspaces of \( E \) as desired in the hypotheses. Therefore, for each \( n \in \mathbb{N} \), there exists a finite rank endomorphism \( u_n \) such that \( \lim_{n \to \infty} u_n(x) = x \), for all \( x \in \bigcup_{n=1}^{\infty} G_n \). Since, \( \sup_{1 \leq n < \infty} \|u_n\| < \infty \) and \( \bigcup_{n=1}^{\infty} G_n \) is dense in \( E \), \( \lim_{n \to \infty} u_n(x) = x, x \in E \). Then, proceeding as in Theorem 3.2, one may conclude that \( E \) has an approximative atomic decomposition.

**Theorem 3.6.** A separable Banach space \( E \) has an approximative atomic decomposition if there exists a constant \( \lambda \geq 1 \) such that for every finite dimensional subspace \( G \) of \( E \) and every \( \delta > 0 \) there exists a finite rank endomorphism \( u \in B(E) \) satisfying

\[
\|u\| \leq \lambda \text{ and } \|u(x) - x\| < \delta\|x\|, \quad x \in G.
\]

**Proof.** Let \( \{y_n\} \) be a sequence in \( E \) such that \( [y_n] = E \). Then, by hypotheses, for each \( n \in \mathbb{N} \) there exists a finite rank endomorphism \( u_n \subset B(E) \) such that

\[
\|u_n\| \leq \lambda, \quad n \in \mathbb{N} \quad \text{and}
\]

\[
\|u_n(x) - x\| < \frac{1}{n}\|x\|, \quad x \in G_n = [y_1, y_2, \ldots, y_n], \quad n \in \mathbb{N}.
\]

Since \( G_n \subset G_{n+1}, n \in \mathbb{N} \) and \( \|u_n\| < \lambda, n \in \mathbb{N} \), it follows that \( \lim_{n \to \infty} u_n(x) = x, x \in E \). Also \( \sup_{1 \leq n < \infty} \|u_n\| < \infty \).

Hence, proceeding as in Theorem 3.2, \( E \) has an approximative atomic decomposition.

In [10], it has been proved that if two Banach spaces have Banach frames, then their product space also has a Banach frame. The following is a similar result regarding approximative atomic decompositions.

**Theorem 3.7.** Let \( E \) and \( F \) be Banach spaces with approximative atomic decompositions. Then the product space \( E \times F \) also has an approximative atomic decomposition.
Proof. Proceeding as in Theorem 3.2, for each \( n \in \mathbb{N} \), \( u_n \) is a continuous linear operator on \( E \) with \( \dim(u_n(E)) < \infty \) and \( v_n \) is a continuous linear operator on \( F \) with \( \dim(v_n(F)) < \infty \) such that

\[
\lim_{n \to \infty} u_n(x) = x, \quad x \in E \quad \text{and} \quad \lim_{n \to \infty} v_n(y) = y, \quad y \in F.
\]

For each \( n \in \mathbb{N} \), define \( h_n : E \times F \to E \times F \) by

\[
h_n(x, y) = (u_n(x), v_n(y)), \quad (x, y) \in E \times F.
\]

Then each \( h_n \) is well defined continuous linear operator on \( E \times F \) with \( \dim(h_n(E \times F)) < \infty \) and

\[
\lim_{n \to \infty} h_n(x, y) = (x, y), \quad (x, y) \in E \times F.
\]

Therefore, again as in Theorem 3.2, \( E \times F \) has an approximative atomic decomposition.

Finally, as an application of approximative atomic decompositions, we have the following corollary

Corollary 3.8. If \( E \) and \( F \) are Banach spaces having bounded approximation property, then the product space \( E \times F \) also has bounded approximation property.

Proof. Follows in view of Theorem 3.3 and Theorem 3.7.

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References


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