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On Weighted Banach Frames

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Abstract. We introduce and study weighted Banach frames in Banach spaces. Necessary and/or sufficient conditions for a weighted Banach frame to be exact are given. An application of weighted Banach frames is discussed.

1. Introduction

Frames for Hilbert spaces were introduced by Duffin and schaeffer in [8] in 1952, while addressing some deep problems in non-harmonic Fourier series. Later, in 1986, Daubechies, Grossmann and Meyer [7] found new applications to wavelets and Gabor transforms in which frames played an important role.

Today, frames play important roles in many applications in mathematics, science and engineering. In particular frames are widely used in sampling theory, wavelet theory, wireless communication, signal processing, image processing, differential equations, filter banks, geophysics, quantum computing, wireless sensor network, multiple-antenna code design and many more. Reason is that frames provide both great liberties in design of vector space decompositions, as well as quantitative measure on computability and robustness of the corresponding reconstructions. In the theoretical direction, powerful tools from operator theory and Banach spaces are being employed to study frames. For a nice and comprehensive survey on various types of frames, one may refer to [1, 5] and the references therein.

Coifman and Weiss [6] introduced the notion of atomic decomposition for function spaces. Later, Feichtinger and Grochenig [10] extended this idea to Banach spaces. This concept was further is generalized by Grochenig [13] who introduced the notion of Banach frames for Banach Spaces. Casazza, Han and Larson [2] also carried out a study of atomic decompositions and Banach frames.

In this paper we introduce and study weighted Banach frames in Banach spaces. Necessary and/or sufficient conditions for a weighted Banach frame to be exact are given. We know that if a Banach space has a Banach frame, then it can be

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recovered by pre-frame operator. In Section 4 we discuss a problem of disturbance of a function(signal) and existence of the associated pre-frame operator. A special type of a weighted Banach frame, namely weighted Banach frame of type $\omega\omega P^*$ is introduced and an application of weighted Banach frame is discussed.

2. Preliminaries

Throughout this paper, $\mathscr{X}$ will be denote an infinite dimensional Banach space over the scalar field $\mathbb{K}$ (which will be $\mathbb{R}$ or $\mathbb{C}$) and $\mathscr{X}^*$ the conjugate space of $\mathscr{X}$. For a sequence $\{f_n\} \subset \mathscr{X}^*$, $[f_n]$ denotes the closure of span$\{f_n\}$ in the norm topology of $\mathscr{X}^*$ and $\widetilde{[f_n]}$ the closure of span$\{f_n\}$ in $\sigma(\mathscr{X}^*, \mathscr{X})$-topology. A weight $\omega = \{\omega_n\}$ is a sequence of positive real numbers.

Definition 2.1 ([13]). Let $\mathscr{X}$ be a Banach space and let $\mathscr{X}_d$ be an associated Banach space of scalar valued sequences indexed by $\mathbb{N}$. Let $\{f_n\} \subset \mathscr{X}^*$ and $S : \mathscr{X}_d \to \mathscr{X}$ be given. The pair $(\{f_n\}, S)$ is called a *Banach frame* for $\mathscr{X}$ with respect to $\mathscr{X}_d$ if:

(i) $\{f_n(x)\} \in \mathscr{X}_d$, for each $x \in \mathscr{X}$.

(ii) There exist positive constants A and B with $0 < A \le B < \infty$ such that

$$A\|x\|_{\mathscr{X}} \le \|\{f_n(x)\}\|_{\mathscr{X}_d} \le B\|x\|_{\mathscr{X}}, \quad \text{for all } x \in \mathscr{X}. \tag{2.1}$$

(iii) S is a bounded linear operator such that

$$S(\{f_n(x)\}) = x, \quad \text{for all } x \in \mathscr{X}.$$

The positive constants A and B are called *lower* and *upper frame bounds* of the Banach frame $(\{f_n\}, S)$, respectively. The operator $S : \mathscr{X}_d \to \mathscr{X}$ is called the *reconstruction operator* (or the *pre-frame operator*). The inequality (2.1) is called the *frame inequality*.

The Banach frame $(\{f_n\}, S)$ is called *tight* if $A = B$ and *normalized tight* if $A = B = 1$. If removal of one f_n renders the collection $\{f_n\} \subset \mathscr{X}^*$ no longer a Banach frame for $\mathscr{X}$, then $(\{f_n\}, S)$ is called an *exact Banach frame*.

Lemma 2.2. *Let $\mathscr{X}$ be a Banach space and $\{f_n\} \subset \mathscr{X}^*$ be a sequence such that $\{x \in \mathscr{X} : f_n(x) = 0, n \in \mathbb{N}\} = \{0\}$. Then $\mathscr{X}$ is linearly isometric to the Banach space $X = \{\{f_n(x)\} : x \in \mathscr{X}\}$, where the norm is given by $\|\{f_n(x)\}\|_X = \|x\|_{\mathscr{X}}$, $x \in \mathscr{X}$.*

Lemma 2.3 ([15]). *Let $(\{f_n\}, S)$ (where $\{f_n\} \subset \mathscr{X}^*$, $S : \mathscr{X}_d \to \mathscr{X}$) be a Banach frame for $\mathscr{X}$ with respect to $\mathscr{X}_d$. Then $(\{f_n\}, S)$ is exact if and only if $f_n \notin [\widetilde{f_i}]_{i \neq n}$ for all $n \in \mathbb{N}$.*

Proof. Suppose that $(\{f_n\}, S)$ is exact. Fix $n \in \mathbb{N}$. Then, there exists no reconstruction operator $\widetilde{S}$ such that $(\{f_i\}_{i \neq n}, \widetilde{S})$ is a Banach frame for $\mathscr{X}$. Therefore, by using Lemma 2.2, $\widetilde{[f_i]}_{i \neq n} \neq E^*$. Hence $f_n \notin \widetilde{[f_i]}_{i \neq n}$, for all $n \in \mathbb{N}$.

Conversely, let $f_n \notin \widetilde{[f_i]}_{i \neq n}$, for all $n \in \mathbb{N}$ and let $(\{f_n\}, S)$ be not exact. Then, there exists a positive integer m_0 and a reconstruction operator $\widetilde{S}_0$ such that $(\{f_i\}_{i\neq m_0}, \widetilde{S_0})$ is a Banach frame for $\mathscr{X}$. Thus, by using frame inequality for $(\{f_i\}_{i\neq m_0}, \widetilde{S_0})$, we obtain $\widetilde{[f_i]}_{i\neq m_0} = \mathscr{X}^*$. This gives $f_{m_0} \in \widetilde{[f_i]}_{i\neq m_0}$, a contradiction. $\square$

Remark 2.4. If $(\{f_n\}, S)$ is exact, then by Lemma 2.3 there exists a sequence $\{x_n\} \subset \mathscr{X}$, called *admissible sequence* of vector to $(\{f_n\}, S)$ such that

$$f_i(x_j) = \delta_{i,j} = \begin{cases} 1, & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \quad \text{for all } i, j \in \mathbb{N}.$$

3. Weighted Banach Frames

Definition 3.1. Let $\omega = \{\omega_n\}$ be a weight. A pair $(\{f_n\}, \mathfrak{S})$ $(\{f_n\} \subset \mathscr{X}^*, \mathfrak{S} : \mathscr{X}_d \to \mathscr{X})$ is called a *weighted Banach frame* (or *ω-Banach frame*) for $\mathscr{X}$ with respect to $\mathscr{X}_d$ if:

(i) $\{\omega_n f_n(x)\} \in \mathscr{X}_d$, for each $x \in \mathscr{X}$.

(ii) There exist positive constants C and D $(0 < \mathrm{C} \leq \mathrm{D} < \infty)$ such that

$$\mathrm{C}\|x\|_{\mathscr{X}} \leq \|\{\omega_n f_n(x)\}\|_{\mathscr{X}_d} \leq \mathrm{D}\|x\|_{\mathscr{X}}, \quad \text{for all } x \in \mathscr{X}. \tag{3.1}$$

(iii) S is a bounded linear operator such that

$$\mathfrak{S}(\{\omega_n f_n(x)\}) = x, \quad \text{for all } x \in \mathscr{X}.$$

The positive constants C and D are called *lower* and *upper frame bounds* of the weighted Banach frame $(\{\omega_n f_n\}, \mathfrak{S})$, respectively. The operator $\mathfrak{S} : \mathscr{X}_d \to \mathscr{X}$ is called the *reconstruction operator* (or the *pre-frame operator*). The inequality (3.1) is called the *weighted frame inequality*.

The weighted Banach frame $(\{\omega_n f_n\}, \mathfrak{S})$ is called *tight* if $\mathrm{C} = \mathrm{D}$ and *normalized tight* if $\mathrm{C} = \mathrm{D} = 1$. If there exists no reconstruction operator $\mathfrak{S}_0$ such that $(\{\omega_n f_n\}_{n\neq j}, \mathfrak{S}_0)$ is a weighted Banach frame for $\mathscr{X}$ $(j \in \mathbb{N})$, then $(\{\omega_n f_n\}, \mathfrak{S})$ is called an *exact weighted Banach frame*.

Remark 3.2. Let $\mathscr{X} = l^\infty$ and $\mathscr{X}_d = l^\infty$ Define $\{f_n\} \subset \mathscr{X}^*$ by $f_n(x) = \xi_n$ for all $n \in \mathbb{N}, x = \{\xi_n\} \in \mathscr{X}$. Then there exists a reconstruction operator $S : \mathscr{X}_d \to \mathscr{X}$ such that $(\{f_n\}, S)$ is Banach frame for $\mathscr{X}$ with respect to $\mathscr{X}_d$. Let $\omega = \{\omega_n = \frac{1}{n}\}$ be a weight. Then, there exists a reconstruction operator $\mathfrak{S} : \mathscr{X}_d \to \mathscr{X}$ such that $(\{\omega_n f_n\}, \mathfrak{S})$ is a weighted Banach frame for $\mathscr{X}$ with respect to $\mathscr{X}_d$. But for the weight $\omega_0 = \{\omega_n^* = n\}$ there exists no reconstruction operator $\mathfrak{S}_o : \mathscr{X}_d \to \mathscr{X}$ such that $(\{\omega_n^* f_n\}, \mathfrak{S}_o)$ is a weighted Banach frame for $\mathscr{X}$ with respect to $\mathscr{X}_d$. However, there are other sequence spaces with respect to which $(\{\omega_n^* f_n\}, \mathfrak{S}_o)$ form a weighted Banach frame for $\mathscr{X}$. Thus, various type of sequence spaces involved in the definition of weighted Banach frame. This leads to the study of relation between various types of sequence spaces through theory of weighted Banach frames.

Example 3.3. Let $\mathscr{X} = l^\infty$. The weighted Banach frame $(\{\omega_n f_n\}, \mathfrak{S})$ with weight $\omega = \{\omega_n = \frac{1}{n}\}$ given in Remark 3.2 is exact and tight weighted Banach frame for $\mathscr{X}$ with respect to $\mathscr{X}_d = \{\{\omega_n f_n(x)\} : x \in \mathscr{X}\}$.

Now consider the weight$\omega = \{\omega_n\}$, where $\omega_3 = \frac{1}{2}$ and $\omega_n = 1$, $n \neq 3$. Define $\{f_n\} \subset \mathscr{X}^*$ by $f_1(x) = \xi_1$, $f_{n+1}(x) = \xi_{n-1}$ $(n = 2,3,4,\ldots)$, $x = \{\xi_j\} \in \mathscr{X}$. Then, there exists a reconstruction operator $\widetilde{\mathfrak{S}}$ such that $(\{\omega_n f_n\}, \widetilde{\mathfrak{S}})$ is a non-exact weighted Banach frame for $\mathscr{X}$ with respect to $\mathscr{X}_{d_0} = \{\{\omega_n f_n(x)\} : x \in \mathscr{X}\}$.

The first proposition of this section provides necessary and sufficient conditions for exactness of a weighted Banach frame.

Proposition 3.4. *Let $\omega = \{\omega_n\}$ be a weight and let $(\{\omega_n f_n\}, \mathfrak{S})$ be a weighted Banach frame for $\mathscr{X}$ with respect to $\mathscr{X}_d$. Then $(\{\omega_n f_n\}, \mathfrak{S})$ is exact if and only if $\omega_n f_n \notin [\widetilde{\omega_i f_i}]_{i \neq n}$ for all $n \in \mathbb{N}$.*

Proof. If $(\{\omega_n f_n\}, \mathfrak{S})$ is an exact weighted Banach frame, then there exists no reconstruction operator $\mathfrak{S}_o$ such that $(\{\omega_n f_n\}_{n \neq i}, \mathfrak{S}_o)$ is a weighted Banach frame for $\mathscr{X}$. By using weighted frame inequality of $(\{\omega_n f_n\}, \mathfrak{S})$ we conclude that $\omega_n f_n \notin [\widetilde{\omega_n f_i}]_{i \neq n}$ for all $n \in \mathbb{N}$. Conversely, assume that $\omega_n f_n \notin [\widetilde{\omega_i f_i}]_{i \neq n}$ for all $n \in \mathbb{N}$. If $(\{\omega_n f_n\}, \mathfrak{S})$ is not exact, then there exists a reconstruction operator $\widetilde{\mathfrak{S}}$ such that $(\{\omega_n f_n\}_{n \neq k}, \widetilde{\mathfrak{S}})$ (for some k) is a weighted Banach frame for $\mathscr{X}$. By weighted frame inequality of $(\{\omega_n f_n\}_{n \neq k}, \widetilde{\mathfrak{S}})$ we have $\omega_n f_n \in [\widetilde{\omega_i f_i}]_{i \neq n}$, a contradiction. This completes the proof. $\square$

Remark 3.5. A weighted Banach frame $(\{\omega_n f_n\}, \mathfrak{S})$ for $\mathscr{X}$ is exact if there is a sequence $\{x_n\} \subset \mathscr{X}$ such that

$$\lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i^n \omega_i f_i = f \quad \Rightarrow \quad \lim_{n\to\infty} \alpha_i^n = f(x_i), \quad i \in \mathbb{N}.$$

Now we give necessary condition under which a weighted Banach frame turn out to be exact.

Proposition 3.6. *If $(\{\omega_n f_n\}, \mathfrak{S})$ is an exact weighted Banach frame for $\mathscr{X}$ with admissible sequence $\{x_n\} \subset \mathscr{X}$ $([x_n] = \mathscr{X})$, then*

$$\lim_{n\to\infty} \alpha_i^n = 0,\ i \in \mathbb{N} \quad \Rightarrow \quad \lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i^n x_i = 0\,.$$

Proof. For each $x = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{m_n} \alpha_i^n x_i$, we have

$$\omega_j f_j(x) = \omega_j f_j \left(\lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i^n x_i \right) = \lim_{n\to\infty} \alpha_j^{(n)} = 0, \quad j \in \mathbb{N}\,.$$

Therefore, by weighted frame inequality for $(\{\omega_n f_n\}, \mathfrak{S})$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{m_n} \alpha_i^n x_i = 0$. This complete the proof. $\square$

Let $(\{\omega_n f_n\}, \mathfrak{S})$ be an exact weighted Banach frame for $\mathscr{X}$ and that $\sup_n \| \sum_{i=1}^{n} \alpha_i \omega_i f_i \| < \infty$. Then, the scalars $\alpha' s$ are determined in the following proposition.

Proposition 3.7. *Let $(\{w_n f_n\}, \mathfrak{S})$ be an exact weighted Banach frame for $\mathscr{X}$ with respect to $\mathscr{X}_d$ with admissible sequence $\{x_n\} \subset \mathscr{X}$,such that $\gamma_{[x_n]}[w_n f_n] > 0$ and $\sup_{1\leq n\leq\infty} \left\| \sum_{i=1}^{n} \alpha_i w_i f_i \right\| < \infty$. Then, there exists an $\phi \in \mathscr{X}^*$ such that $\phi(x_n) = \alpha_n$, for all $n \in \mathbb{N}$.*

Proof. Let w be the canonical embedding of $[w_n f_n]$ into $[\pi(x_i)]^*$, where $\pi : \mathscr{X} \to \mathscr{X}^{**}$ is canonical embedding of $\mathscr{X}$ into $\mathscr{X}^{**}$. Since $[\pi(x_n)]$ is separable, the $\sigma([\pi(x_n)]^*[\pi(x_n)])$-topology is metrizable on bounded sets of $[\pi(x_i)]^*$. Thus there exists a subsequence

$$\left\{ \sum_{i=1}^{n_k} \alpha_i w(w_i f_i) \right\}_k \text{ of } \left\{ \sum_{i=1}^{n} \alpha_i w(w_i f_i) \right\}_n$$

which is $\sigma([\pi(x_n)]^*[\pi(x_n)])$-convergent to some $\psi \in [\pi(x_i)]^*$.
Since $\gamma_{[x_n]}[w_n f_n] > 0$, w is an isomorphism of $[w_n f_n]$ onto $[\pi(x_i)]^*$. Therefore, there exists a $\phi \in [w_n f_n]$ such that $\psi = w(\phi)$. Hence

$$\begin{aligned}
\phi(x_n) &= (w(f))(\pi(x_n)) \\
&= \psi(\pi(x_n)) \\
&= \sigma([\pi(x_n)]^*[\pi(x_n)]) - \lim_{n\to\infty} \sum_{i=1}^{n_k} \alpha_i((w_i f_i))(\pi(x_n)) \\
&= \alpha_n, \quad n \in \mathbb{N}.
\end{aligned}$$

□

The following proposition shows that the action of a bounded linear operator on a weighted Banach frame is a weighted Banach frame with respect to same weight.

Proposition 3.8. *Let $(\{w_n f_n\}, \mathfrak{S})$ be an exact weighted Banach frame for $\mathscr{X}$ with respect to $\mathscr{X}_d$. Let T be a continuous linear operator from $\mathscr{X}$ onto another Banach space $\mathscr{W}$ such that $T^*(w_n g_n) = w_n f_n$, $n \in \mathbb{N}$. Then, there exists a reconstruction operator $\mathfrak{U}$ such that $(\{w_n g_n\}, \mathfrak{U})$ is a weighted Banach frame for $\mathscr{W}$. Moreover, if $(\{w_n f_n\}, \mathfrak{S})$ is exact, then $(\{w_n g_n\}, \mathfrak{U})$ is also exact.*

Proof. For each $y \in \mathscr{W}$, there exists an $x \in \mathscr{X}$, such that $T(x) = y$. Let $(w_n g_n) y = 0$, for all $n \in \mathbb{N}$. Then we have

$$(w_n f_n)(x) = T^*(w_n g_n)(x) = (w_n g_n)(Tx) = (w_n g_n) y = 0, \text{ for all } n \in \mathbb{N}.$$

By weighted Banach frame inequality for $(\{w_n f_n\}, \mathscr{W})$, this gives $y = 0$. Hence by Lemma 2.2 there exists a reconstruction operator $\mathfrak{U}$ such that $(\{w_n g_n\}, \mathfrak{U})$ is a weighted Banach frame for $\mathscr{W}$ with respect to $\mathscr{W}_{d_0} = \{\{(w_n g_n)(z)\} : z \in F\}$. If

$(\{w_nf_n\}, \mathfrak{S})$ is exact, then there exists a sequence $\{v_n\} \subset E$, such that $(w_nf_n)(v_n) = \delta_{n,m}$, for all $n \in \mathbb{N}$.
Therefore

$$\begin{aligned}(w_ng_n)(Tv_m) &= T^*(w_ng_n)v_m \\ &= (w_nf_n)(v_m) \\ &= \delta_{n,m}, \quad \text{for all } n, m \in \mathbb{N}.\end{aligned}$$

Thus, $w_ng_n \notin [w_ig_i]_{i\neq n}$ for all $n \in \mathbb{N}$. Hence by Proposition 3.4 $(\{w_ng_n\}, \mathfrak{U})$ is an exact weighted Banach frame for $\mathscr{W}$. □

To conclude the paper we show that weighted Banach frames are invariant under linear homeomorphism. The proof of following proposition uses certain ideas developed in [11].

Proposition 3.9. *Let $\mathscr{X}$ and $\mathscr{W}$ be Banach spaces and let $(\{w_nf_n\}, \mathfrak{S})$ be a weighted Banach frame for $\mathscr{X}$ with respect to $\mathscr{X}_d$ and with best bounds A_1, B_1. Let $\mathfrak{U} : \mathscr{X} \to \mathscr{W}$ be a linear homeomorphism. Then, $(\{(\mathfrak{U}^{-1})^*(w_nf_n\}), \mathfrak{U}\mathfrak{S})$ is a weighted Banach frame for $\mathscr{W}$ with respect to $\mathscr{X}_d$ and its best bounds A_2, B_2 satisfy the inequalities*

$$A_1\|\mathfrak{U}\|^{-1} \le A_2 \le A_1\|\mathfrak{U}^{-1}\| \quad \text{and} \quad B_1\|\mathfrak{U}\|^{-1} \le B_2 \le B_1\|\mathfrak{U}^{-1}\|.$$

Proof. By hypothesis

$$A_1\|x\|_{\mathscr{X}} \le \|\{(w_nf_n)(x)\}\|_{\mathscr{X}_d} \le B_1\|x\|_{\mathscr{X}}, \quad \text{for all } x \in \mathscr{X}. \tag{3.2}$$

Let $y \in \mathscr{W}$ be arbitrary. Then

$$\begin{aligned}\|y\|_{\mathscr{W}} &= \|\mathfrak{U}\mathfrak{U}^{-1}y\|_{\mathscr{W}} \\ &\le \|\mathfrak{U}\|\|\mathfrak{U}^{-1}y\|_{\mathscr{X}}.\end{aligned}$$

Then, by using (3.2), we have

$$\begin{aligned}A_1\|\mathfrak{U}\|^{-1}\|y\|_{\mathscr{W}} &\le A_1\|\mathfrak{U}^{-1}y\|_{\mathscr{X}} \\ &\le \|\{(w_nf_n)(\mathfrak{U}^{-1}y)\}\|_{\mathscr{X}_d} \\ &\le B_1\|\mathfrak{U}^{-1}y\|_{\mathscr{W}} \\ &\le B_1\|\mathfrak{U}^{-1}\|\|y\|_{\mathscr{W}}.\end{aligned}$$

Now $\{(\mathfrak{U}^{-1})^*(w_nf_n)(y)\} \in \mathscr{X}_d$, for all $y \in \mathscr{W}$ and

$$\begin{aligned}\mathfrak{U}\mathfrak{S}\{(\mathfrak{U}^{-1})^*(w_nf_n)(y)\} &= \mathfrak{U}\mathfrak{S}\{(w_nf_n)(\mathfrak{U}^{-1}y)\} \\ &= \mathfrak{U}(\mathfrak{U}^{-1}y) \\ &= y, \quad \text{for all } y \in \mathscr{W}.\end{aligned}$$

Hence $(\{(\mathfrak{U}^{-1})^*(w_n f_n)\}, \mathfrak{U}\mathfrak{S})$ is a weighted Banach frame for $\mathcal{W}$ with one of choice of bounds $A_1\|\mathfrak{U}\|^{-1}$ and $B_1\|\mathfrak{U}^{-1}\|$. If A_2, B_2 are the best bounds for $(\{(\mathfrak{U}^{-1})^*(w_n f_n\}), \mathfrak{U}\mathfrak{S})$, then

$$A_1\|\mathfrak{U}\|^{-1} \leq A_2 \quad \text{and} \quad B_2 \leq B_1\|\mathfrak{U}^{-1}\|. \tag{3.3}$$

Now for all $x \in \mathcal{X}$, we have

$$\begin{aligned}\|x\|_{\mathcal{X}} &= \|\mathfrak{U}^{-1}\mathfrak{U}\| \\ &\leq \|\mathfrak{U}^{-1}\|\|\mathfrak{U}x\|_{\mathcal{W}}.\end{aligned}$$

Therefore

$$\begin{aligned}A_2\|\mathfrak{U}^{-1}\|^{-1}\|x\|_{\mathcal{X}} &\leq A_2\|\mathfrak{U}x\|_{\mathcal{W}} \\ &\leq \|(\mathfrak{U}^{-1})^*(w_n f_n)(\mathfrak{U}x)\|_{\mathcal{X}_d} \\ &= (\|\{(w_n f_n)(x)\}\|_{\mathcal{X}_d}) \\ &\leq B_2\|\mathfrak{U}x\|_{\mathcal{W}} \\ &\leq B_2\|\mathfrak{U}\|\|x\|_{\mathcal{X}}.\end{aligned}$$

By using the fact that A_1, B_1 are the best bounds for $(\{w_n f_n\}, \mathfrak{S})$ we obtain:

$$A_2\|\mathfrak{U}^{-1}\| \leq A_1 \quad \text{and} \quad B_1 \leq B_2\|\mathfrak{U}\|. \tag{3.4}$$

Hence by (3.3) and (3.4), we have

$$A_1\|\mathfrak{U}\|^{-1} \leq A_2 \leq A_1\|\mathfrak{U}^{-1}\| \quad \text{and} \quad B_1\|\mathfrak{U}\|^{-1} \leq B_2 \leq B_1\|\mathfrak{U}^{-1}\|.$$

This completes the proof. □

Corollary 3.10. *Let $(\{w_n f_n\}, \mathfrak{S})$ be an exact weighted Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$. Then, $(\{(\mathfrak{U}^{-1})^*(w_n f_n)\}, \mathfrak{U}\mathfrak{S})$ is also an exact weighted Banach frame for $\mathcal{W}$.*

4. Weighted Banach frames of Type $\omega\omega P^*$

Recently, Banach frames of type ωP^* introduced and studied in [20].

Definition 4.1 ([20]). A Banach frame $(\{f_n\}, S)$ for a Banach space $\mathcal{X}$ is said to be of type ωP^* (*weak of type* P^*) if there exists a functional $\Phi \in E^{**}$ such that

$$\Phi(f_n) = 1, \quad \text{for all } n \in \mathbb{N}.$$

The functional Φ is called an *associated functional* of $(\{f_n\}, S)$.

Imagine a disturbed signal in space which was recovered by a pre-frame operator associated with a Banach frame before its transmission to a receiver. That is, consider a signal space $\mathcal{X}_0 = L^2(\Omega)$ (say). Let $(\{\phi_n\}, \widetilde{\mathfrak{V}})$ be a Banach frame for $\mathcal{X}_0$. Then, elements of $\mathcal{X}_0$ can be recovered by pre-frame operator $\widetilde{\mathfrak{V}}$. Let $0 \neq f_0 \in L^2(\Omega)$. Then, in general, there is no pre-frame operator $\mathfrak{U}$ associated with

$\{\phi_n + f_0\}$ which recover $\mathscr{X}_0$. Note that $\{\phi_n + f_0\}$ corresponds to components of disturbed signal, i.e. it is associated to analysis operator of concern Banach frame. We say that f_0 is a *doping functional*. We extend this doping problem regarding recovery of concern signal space to general Banach spaces. That is, if $(\{f_n\}, S)$ is a Banach frame for a Banach space $\mathscr{X}$, then $\mathscr{X}$ (in general) not recovered by pre-frame operator $\widetilde{\mathfrak{U}_o}$ associated with $\{f_n + f_0\}$. However, we can recover $\mathscr{X}$ by mean of $\{f_n + f_0\}$, provided $(\{f_n\}, S)$ is of type ωP^* with a certain action of associated functional of $(\{f_n\}, S)$ on f_0. A proposition in this direction is given in [20]

Proposition 4.2 ([20]). *Let $(\{f_n\}, S)$ be a Banach frame of type ωP^* for a Banach space $\mathscr{X}$ with associated functional Φ. Then, there exists a reconstruction operator $\mathfrak{U}$ such that $(\{f_n + f_0\}, \mathfrak{U})$ is Banach frame for $\mathscr{X}$ provided $\Phi(f_0) \neq -1$, where f_0 is a doping functional.*

Following example gives an application of Proposition 4.2.

Example 4.3. Let $\mathscr{X} = c_0$. Define $\{f_n\} \subset \mathscr{X}^*$ by

$$\left.\begin{array}{l} f_1(x) = 2\xi_1 \\ f_n(x) = \xi_n, \quad n = 2, 3, \ldots \end{array}\right\}, \quad x = \{\xi_j\} \in \mathscr{X}\,.$$

Then, there exists a reconstruction operator $S_0 : \mathscr{X}_{d_0} = \{\{f_n(x)\} : x \in \mathscr{X}\} \to \mathscr{X}$ such that $(\{f_n\}, S_0)$ is a Banach frame of type ωP^* for $\mathscr{X}$ with associated functional $\Phi = (1/2, 1, 1, 1, \ldots) \in \mathscr{X}^{**}$. Consider the doping functional $f_0 = f_2$. Then, $f_0 \in \mathscr{X}^*$ is a non-zero functional such that $\Phi(f_0) \neq -1$. Hence by Proposition 4.2 there exists a reconstruction operator $\mathfrak{U}$ such that $(\{f_n + f_0\}, \mathfrak{U})$ is a Banach frame for $\mathscr{X}$. Thus, $\mathscr{X}$ is recovered by a pre-frame operator $\mathfrak{U}$ associated with $(\{f_n + f_0\}$.

Problem. What happen if doping functional in Example 4.3 is $f_0 = -f_3$? This problem makes sense, because disturbances are not constant!

The answer to this problem is partial negative. In this case $\Phi(f_0) = -1$ and so Proposition 4.2 does not work. However, if we give some suitable weight to $\{f_n\}$, then we can recover $\mathscr{X}$. This is a motivation for the existence of some special types of weighted Banach frames regarding recovery of concern Banach space $\mathscr{X}$. This is given in the following definition.

Definition 4.4. Let $\omega = \{\omega_n\}$ be a weight. A weighted Banach frame $(\{\omega_n f_n\}, \mathfrak{S})$ for a Banach space $\mathscr{X}$ is said to be of type $\omega\omega P^*$ (*weighted weak of type P^**) if there exists a functional $\Phi_0 \in E^{**}$ such that

$$\Phi_0(\omega_n f_n) = 1, \quad \text{for all } n \in \mathbb{N}.$$

The functional Φ_0 is called an *associated functional* of $(\{\omega_n f_n\}, \mathfrak{S})$.

Remark 4.5. The condition $\Phi_0(\omega_n f_n) = 1$, $n \in \mathbb{N}$, resembles dynamics of frames! Physical interpretation of this can be understood as the earth rotates about its axis. Here 1 is the axis and Φ_0 is the action of rotation on $\{\omega_n f_n\}$.

The following proposition reflects importance of weighted Banach frames for Banach spaces. However, it is a generalization of Proposition 4.2.

Proposition 4.6. *Let* $(\{\omega_n f_n\}, \mathfrak{S})$ *be a Banach frame of type* $\omega\omega P^*$ *for a Banach space* $\mathscr{X}$ *with associated functional* Φ_0. *Then, there exists a reconstruction operator* $\mathfrak{U}_o$ *such that* $(\{(\omega_n f_n)+f_0\}, \mathfrak{U}_o)$ *is Banach frame for* $\mathscr{X}$ *provided* $\Phi_0(f_0) \neq -1$, *where* f_0 *is a doping functional.*

Proof. Similar to proof of Proposition 4.2. □

5. Conclusion

Consider the Banach frame $(\{f_n\}, S_0)$ given in Example 4.3 and doping functional $f_0 = f_2$. Then, by Proposition 4.2 there exists a reconstruction operator $\mathfrak{U}$ such that $(\{f_n + f_0\}, \mathfrak{U})$ is a Banach frame for $\mathscr{X}$.
Since disturbances are not constant! That is, if we consider doping functional $f_0 = -f_3$, then there is no pre-frame operator associated with $\{f_n + f_0\}$ which recover $\mathscr{X}$. However, if we give suitable weight to $\{f_n\}$, in this case, consider the weight $\omega = \{n\}$, then associated functional Φ_0 of the corresponding weighted Banach frame $(\{\omega_n f_n\}, \mathfrak{V})$ satisfies $\Phi_0(f_0) \neq -1$. Hence by Proposition 4.6, there exists a reconstruction operator $\mathfrak{U}_o$ such that $(\{\omega_n f_n+f_0\}, \mathfrak{U}_o)$ is a Banach frame for $\mathscr{X}$. So, $\mathscr{X}$ is recovered by pre-frame operator $\mathfrak{U}_o$ associated with $(\{\omega_n f_n+f_0\}$. Thus, Proposition 4.6 shows that how weighted Banach frame are useful in recovery of a function(signal) in general Banach spaces.

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