The Quasi-Hyperbolic Tribonacci and Quasi-Hyperbolic Tribonacci-Lucas Functions

Dursun Ta¸sci and Huriye Azman*

Department of Mathematics, Gazi University, Faculty of Science, Teknikokullar, 06500 Ankara, Turkey

*Corresponding author: huriyeazman@gazi.edu.tr

Abstract. In the present paper, we studied an extension of the classical hyperbolic functions. We wrote a new relation that is equal to the Binet formula of the Tribonacci-Lucas numbers. We defined the quasi-hyperbolic Tribonacci and quasi-hyperbolic Tribonacci-Lucas functions. Finally, we investigated the recurrence and hyperbolic properties of these new hyperbolic functions.

Keywords. Hyperbolic functions; Binet's formula; Tribonacci numbers; Tribonacci-Lucas numbers

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1. Introduction

Spickerman defined the Binet’s formula for Tribonacci sequence [5]. In [4], Stakhov and Rozin introduced the symmetrical hyperbolic functions. Then, Falcon and Plaza defined a new class of hyperbolic functions called $k$-Fibonacci hyperbolic functions [2]. Also, Kocer, Tuglu and Stakhov obtained the hyperbolic functions with second order recurrence sequences [3].

In this paper, we studied, in a sense, an extension of the classical hyperbolic functions. Our main goal is to get the continuous versions of Tribonacci and Tribonacci-Lucas numbers. For this, by using the same way with Spickerman [5], we wrote a new relation that is equal to the Binet formula of the Tribonacci-Lucas numbers. We defined the quasi-hyperbolic Tribonacci and quasi-hyperbolic Tribonacci-Lucas functions which we named it as quasi-hyperbolic because of it is non-hyperbolic, but it provides some properties of the classical hyperbolic functions. Finally, we investigated the hyperbolic and recurrence properties of these new functions.
1.1 The Tribonacci and Tribonacci-Lucas numbers

The Tribonacci numbers \(\{U_n\}_{n \in \mathbb{N}} = \{0, 1, 1, 2, 4, 7, 13, 24, \ldots\}\) is defined by [1]

\[ U_{n+1} = U_n + U_{n-1} + U_{n-2}, \quad n \geq 2 \tag{1.1} \]

for the initial conditions

\[ U_0 = 0, \quad U_1 = U_2 = 1. \]

The Tribonacci-Lucas numbers \(\{V_n\}_{n \in \mathbb{N}} = \{3, 1, 3, 7, 11, 21, \ldots\}\) is defined by [1]

\[ V_{n+1} = V_n + V_{n-1} + V_{n-2}, \quad n \geq 2 \tag{1.2} \]

with the initial conditions

\[ V_0 = 3, \quad V_1 = 1, \quad V_3 = 3. \]

The characteristic equation of these recurrences (1.1) and (1.2) is

\[ x^3 - x^2 - x - 1 = 0. \tag{1.3} \]

This characteristic equation (1.3) has three roots

\[ x_1 = \rho = \frac{1}{3} \left( \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}} + 1 \right), \tag{1.4} \]

\[ x_2 = \sigma = \frac{1}{6} \left( 2 - \sqrt[3]{19 + 3\sqrt{33}} - i \sqrt{3} \left( \frac{1}{2} \sqrt[3]{19 + 3\sqrt{33}} - \sqrt[3]{19 - 3\sqrt{33}} \right) \right), \]

\[ x_3 = \omega = \bar{\sigma}. \]

The Binet formula for the Tribonacci numbers is

\[ U_n = \frac{\rho^{n+2}}{(\rho - \sigma)(\rho - \omega)} + \frac{\sigma^{n+2}}{(\sigma - \rho)(\sigma - \omega)} + \frac{\omega^{n+2}}{(\omega - \rho)(\omega - \sigma)}. \tag{1.5} \]

In [5], Spickerman defined a new representation which is equivalent to the Binet formula (1.5) after some operations

\[ U_n = \lambda \rho^n + r^n (\psi \cos n \theta + \gamma \sin n \theta) \tag{1.6} \]

where the constants \(\lambda, \rho, r, \psi, \gamma, \theta\) have the following approximate values

\[ \lambda = 0.6184, \quad \rho = 1.8393, \]

\[ r = 0.7374, \quad \psi = 0.3816, \]

\[ \gamma = 0.0374, \quad \theta = 124.69^\circ \]

and the Binet formula for the Tribonacci-Lucas numbers is

\[ V_n = \rho^{n+2} + \sigma^{n+2} + \omega^{n+2} \tag{1.7} \]

Now we will follow the same way with Spickerman to have a new relation that is equal to the Binet formula (1.7).
**Definition 1.** By using the relations 

\[ \sigma = r(\cos \theta + i \sin \theta), \]
\[ \sigma^n = r^n(\cos n\theta + i \sin n\theta); \quad \theta = \tan^{-1}(I(\sigma)/R(\sigma)) \]

where \( I(\sigma) \) is the imaginary part of the \( \sigma \), and \( R(\sigma) \) is the real part of the \( \sigma \), we have

\[ V_n = \rho^n \rho^n + r^n \cos n\theta[2r^n(1 - 2\sin^2 \theta)] + r^n \sin n\theta[-4r^n \sin \theta \cos \theta]. \]

Denoting the coefficients of \( \rho^n \), \( r^n \cos n\theta \) and \( r^n \sin n\theta \) respectively by \( \lambda', \psi' \) and \( \gamma' \), we write

\[ V_n = \lambda' \rho^n + r^n(\psi' \cos n\theta + \gamma' \sin n\theta). \quad (1.8) \]

Approximate values for the constants are

\[ \lambda' = 3.383, \quad \rho = 1.8393, \]
\[ r = 0.7374, \quad \psi' = 0.7353, \]
\[ \gamma' = 2.0357, \quad \theta = 124.69^\circ. \]

### 2. New Quasi-hyperbolic Tribonacci Functions

**Definition 2.** Let \( x \) be a real number. We define the quasi-hyperbolic Tribonacci sine and cosine functions \( sTh(x) \) and \( cTh(x) \) by, respectively

\[ sTh(x) := \lambda \rho^x - r^x(\psi \cos x \theta + \gamma \sin x \theta), \quad (2.1) \]
\[ cTh(x) := \lambda \rho^x + r^x(\psi \cos x \theta + \gamma \sin x \theta). \quad (2.2) \]

The graphics of these functions are given in Figure 1.

![Figure 1](image-url) **Figure 1.** The quasi-hyperbolic Tribonacci sine and cosine.

**2.1 Properties of the Quasi-hyperbolic Tribonacci Functions**

Now, we will give some properties about the quasi-hyperbolic Tribonacci functions, which are similar to the classical hyperbolic functions. Let be \( sTh(x) \) and \( cTh(x) \) as in (2.1) and (2.2).

**Proposition 3** (Pythagorean theorem).

\[ (cTh(x))^2 - (sTh(x))^2 = 4\lambda (\rho r)^x[\psi \cos x \theta + \gamma \sin x \theta]. \]
Proof.
\[
[cTh(x)]^2 - [sTh(x)]^2 = [\lambda \rho^x + r^x(\psi \cos x\theta + \gamma \sin x\theta)]^2 - [\lambda \rho^y - r^y(\psi \cos x\theta + \gamma \sin x\theta)]^2
\]
\[
= (\lambda \rho^x)^2 + 2(\lambda \rho^x \cdot r^x(\psi \cos x\theta + \gamma \sin x\theta)) + [r^x(\psi \cos x\theta + \gamma \sin x\theta)]^2 - [(\lambda \rho^x)^2 - 2(\lambda \rho^x \cdot r^x(\psi \cos x\theta + \gamma \sin x\theta)) + [r^x(\psi \cos x\theta + \gamma \sin x\theta)]^2] = 4(\lambda \rho^x)r^x[\psi \cos x\theta + \gamma \sin x\theta] = 4\lambda(\rho r)^x[\psi \cos x\theta + \gamma \sin x\theta].
\]

\[\square\]

**Proposition 4** (De Moivre).
\[
[cTh(x) + sTh(x)]^n = (2\lambda)^{n-1}[[cTh(nx) + sTh(nx)].
\]

**Proof.** If we look at the LHS of the identity, we have
\[
[cTh(x) + sTh(x)]^n = [\lambda \rho^x + r^x(\psi \cos x\theta + \gamma \sin x\theta) + \lambda \rho^x - r^x(\psi \cos x\theta + \gamma \sin x\theta)]^n
\]
\[
= (2\lambda \rho^x)^n
\]
\[
= (2\lambda)^n \rho^{nx}.
\]

If we look at the RHS of the identity, we have
\[
(2\lambda)^{n-1}[[cTh(nx) + sTh(nx)]
\]
\[
= (2\lambda)^{n-1}[\lambda \rho^{nx} + r^{nx}(\psi \cos nx\theta + \gamma \sin nx\theta) + \lambda \rho^{nx} - r^{nx}(\psi \cos nx\theta + \gamma \sin nx\theta)]
\]
\[
= (2\lambda)^{n-1}[2\lambda \rho^{nx}]
\]
\[
= (2\lambda)^n \rho^{nx}.
\]

So the proof is complete. \[\square\]

**Proposition 5** (Sum).
\[
2\psi(cTh(x + y)) = cTh(x) \cdot cTh(y) + sTh(x) \cdot sTh(y) + 2\rho^{x+y}(\lambda \psi - \lambda^2) - 2r^{x+y} \sin x\theta \sin y\theta(\psi^2 - \gamma^2),
\]
\[
2\psi(sTh(x + y)) = sTh(x) \cdot cTh(y) + cTh(x) \cdot sTh(y) + 2\gamma^{x+y} \sin x\theta \sin y\theta(\psi^2 - \gamma^2).
\]

**Proof.** Firstly, let us prove the first identity.
\[
cTh(x) \cdot cTh(y) + sTh(x) \cdot sTh(y) + 2\rho^{x+y}(\lambda \psi - \lambda^2) - 2r^{x+y} \sin x\theta \sin y\theta(\psi^2 - \gamma^2)
\]
\[
= [\lambda \rho^x + r^x(\psi \cos x\theta + \gamma \sin x\theta)][\lambda \rho^y + r^y(\psi \cos y\theta + \gamma \sin y\theta)]
\]
\[
+ [\lambda \rho^x - r^x(\psi \cos x\theta + \gamma \sin x\theta)][\lambda \rho^y - r^y(\psi \cos y\theta + \gamma \sin y\theta)]
\]
\[
+ 2\rho^{x+y}(\lambda \psi - \lambda^2) - 2r^{x+y} \sin x\theta \sin y\theta(\psi^2 - \gamma^2)
\]
Proof. Let us prove the first equation. Let us look at the LHS and RHS of the identity, respectively.

\begin{align*}
= [2\lambda^2 r^{x+y} + 2r^{x+y}\psi^2 \cos x \theta \cos y \theta + 2r^{x+y}\gamma^2 \sin x \theta \sin y \theta \\
+ 2r^{x+y} \beta \gamma \sin x \theta \cos y \theta + 2r^{x+y} \beta \gamma \cos x \theta \sin y \theta ] \\
+ 2\rho^{x+y}(\lambda \psi - \lambda^2) - 2r^{x+y} \sin x \theta \sin y \theta (\psi^2 - \gamma^2) \\
= 2\psi[\lambda \rho^{x+y} + r^{x+y}(\psi \cos x \theta \cos y \theta - \sin x \theta \sin y \theta ) \\
+ \gamma (\sin x \theta \cos y \theta + \cos x \theta \sin y \theta )] \\
= 2\psi[\lambda \rho^{x+y} + r^{x+y}(\psi \cos (x+y) \theta + \gamma \sin (x+y) \theta )] \\
= 2\psi(cTh(x+y)).
\end{align*}

Now, let prove the second identity.

\begin{align*}
sTh(x) \cdot cTh(y) + cTh(x) \cdot sTh(y) + 2r^{x+y} \sin x \theta \sin y \theta (\psi^2 - \gamma^2) \\
= [\lambda \rho^x - r^x(\psi \cos x \theta + \gamma \sin x \theta)] [\lambda \rho^y + r^y(\psi \cos y \theta + \gamma \sin y \theta )] \\
+ [\lambda \rho^x + r^x(\psi \cos x \theta + \gamma \sin x \theta)] [\lambda \rho^y - r^y(\psi \cos y \theta + \gamma \sin y \theta )] \\
+ 2\gamma^{x+y} \sin x \theta \sin y \theta (\psi^2 - \gamma^2) \\
= [2\lambda^2 r^{x+y} - 2r^{x+y}\psi^2 \cos x \theta \cos y \theta - 2r^{x+y}\gamma^2 \sin x \theta \sin y \theta \\
- 2r^{x+y} \beta \gamma \sin x \theta \cos y \theta - 2r^{x+y} \beta \gamma \cos x \theta \sin y \theta ] + 2\gamma^{x+y} \sin x \theta \sin y \theta (\psi^2 - \gamma^2) \\
= 2\psi[\lambda \rho^{x+y} - r^{x+y}(\psi \cos x \theta \cos y \theta - \sin x \theta \sin y \theta ) + \gamma (\sin x \theta \cos y \theta + \cos x \theta \sin y \theta )] \\
= 2\psi[\lambda \rho^{x+y} - r^{x+y}(\psi \cos (x+y) \theta + \gamma \sin (x+y) \theta )] \\
= 2\psi(sTh(x+y)).
\end{align*}

Corollary 6 (Double argument). If we take \( y = x \) in the previous formulas, then we have

\[ 2\psi(cTh(2x)) = [cTh(x)]^2 + [sTh(x)]^2 + 2\rho^{2x}(\lambda \psi - \lambda^2) - 2r^{2x} \sin^2 x \theta (\psi^2 - \gamma^2), \]

\[ 2\psi(sTh(2x)) = 2sTh(x) \cdot cTh(x) + 2\gamma^{2x} \sin^2 x \theta (\psi^2 - \gamma^2). \]

Now we will study the Tribonacci’s properties of the quasi-hyperbolic Tribonacci numbers.

Proposition 7 (Recursive relations).

\[ sTh(x+1) = sTh(x) + sTh(x-1) + sTh(x-2), \]

\[ cTh(x+1) = cTh(x) + cTh(x-1) + cTh(x-2). \]

Proof. Let us prove the first equation. Let look at the LHS and RHS of the identity, respectively

\[ sTh(x+1) = \lambda \rho^{x+1} - r^{x+1}[\psi \cos (x+1) \theta + \gamma \sin (x+1) \theta ] \]

\[ = \lambda \rho^{x+1} - r^{x+1}[\psi \cos x \theta \cos \theta - \sin x \theta \sin \theta ) + \gamma (\sin x \theta \cos \theta + \cos x \theta \sin \theta )] \]

\[ = \lambda \rho^{x+1} - r^{x+1}\psi \cos x \theta \cos \theta + r^{x+1}\psi \sin x \theta \sin \theta - r^{x+1}\gamma \sin x \theta \cos \theta - r^{x+1}\gamma \sin x \theta \cos \theta \]
\[ sTh(x) + sTh(x-1) + sTh(x-2) \\
= [\lambda \rho^x - r^x(\psi \cos x \theta + \gamma \sin x \theta)] + [\lambda \rho^{x-1} - r^{x-1}(\psi \cos(x-1) \theta + \gamma \sin(x-1) \theta)] \\
+ [\lambda \rho^{x-2} - r^{x-2}(\psi \cos(x-2) \theta + \gamma \sin(x-2) \theta)] \\
= \lambda [\rho^x + \rho^{x-1} + \rho^{x-2}] - r^x[\psi \cos x \theta + \gamma \sin x \theta] \\
- r^{x-1}[\psi(\cos x \theta \cos \theta + \sin x \theta \sin \theta) + \gamma(\sin x \theta \cos \theta - \cos x \theta \sin \theta)] \\
- r^{x-2}[\psi(\cos x \theta \cos 2 \theta + \sin x \theta \sin 2 \theta) + \gamma(\sin x \theta \cos 2 \theta - \cos x \theta \sin 2 \theta)] \\
= \lambda [\rho^x + \rho^{x-1} + \rho^{x-2}] - r^x[\psi \cos x \theta(1 + r^{-1} \cos \theta + r^{-2} \cos 2 \theta) - r^x \psi \sin x \theta(r^{-1} \sin \theta)] \\
+ r^x \gamma \cos x \theta (r^{-1} \sin \theta + r^{-2} \sin 2 \theta) \\
= \lambda [\rho^x + \rho^{x-1} + \rho^{x-2}] - r^x[\psi \cos x \theta(1 + r^{-1} \cos \theta + r^{-2} \cos 2 \theta) \\
- r^x \psi \sin x \theta(r^{-1} \sin \theta + r^{-2} \sin 2 \theta) - r^x \gamma \sin x \theta(1 + r^{-1} \cos \theta + r^{-2} \cos 2 \theta) \\
+ r^x \gamma \cos x \theta (r^{-1} \sin \theta + r^{-2} \sin 2 \theta)] \\
\]

By using \(1 + r^{-1} \cos \theta + r^{-2} \cos 2 \theta = r \cos \theta\) and \(r^{-1} \sin \theta + r^{-2} \sin 2 \theta = -r \sin \theta\), we obtain the result which we look for.

\[ \Box \]

### 3. New Quasi-hyperbolic Tribonacci-Lucas Functions

**Definition 8.** Let \(x\) be a real number. We define the quasi-hyperbolic Tribonacci-Lucas sine and cosine functions \(sTLh(x)\) and \(cTLh(x)\) by, respectively

\[
\begin{align*}
\text{sTLh}(x) & := \lambda' \rho^x - r^x(\psi' \cos x \theta + \gamma' \sin x \theta), \\
\text{cTLh}(x) & := \lambda' \rho^x + r^x(\psi' \cos x \theta + \gamma' \sin x \theta).
\end{align*}
\]

(3.1) \( \text{3.2) } \\

The graphics of the quasi-hyperbolic Tribonacci-Lucas sine and cosine are in Figure 2.

![Figure 2](image-url)

**Figure 2.** The quasi-hyperbolic Tribonacci-Lucas sine and cosine.
3.1 Properties of the Quasi-hyperbolic Tribonacci-Lucas Functions

Now, we will give some properties about the quasi-hyperbolic Tribonacci-Lucas functions, which are similar to the classical hyperbolic functions. Let be $sTLh(x)$ and $cTLh(x)$ as in (3.1) and (3.2).

**Proposition 9** (Pythagorean theorem).

$$[cTLh(x)]^2 - [sTLh(x)]^2 = 4\lambda'(\rho r)^x[\psi' \cos x\theta + \gamma' \sin x\theta].$$

**Proof.**

$$[cTLh(x)]^2 - [sTLh(x)]^2$$

$$= [\lambda' \rho^x + r^x(\psi' \cos x\theta + \gamma' \sin x\theta)]^2 - [\lambda' \rho^x - r^x(\psi' \cos x\theta + \gamma' \sin x\theta)]^2$$

$$= (\lambda' \rho^x)^2 + 2(\lambda' \rho^x) r^x(\psi' \cos x\theta + \gamma' \sin x\theta) + [r^x(\psi' \cos x\theta + \gamma' \sin x\theta)]^2$$

$$- [r^x(\psi' \cos x\theta + \gamma' \sin x\theta)]^2$$

$$= 4(\lambda' \rho^x)r^x[\psi' \cos x\theta + \gamma' \sin x\theta]$$

$$= 4\lambda' (\rho r)^x[\psi' \cos x\theta + \gamma' \sin x\theta].$$

**Proposition 10** (De Moivre).

$$[cTLh(x) + sTLh(x)]^n = (2\lambda')^{n-1}[cTLh(nx) + sTLh(nx)].$$

**Proof.** If we look at the LHS of the identity, we have

$$[cTLh(x) + sTLh(x)]^n$$

$$= [\lambda' \rho^x + r^x(\psi' \cos x\theta + \gamma' \sin x\theta) + \lambda' \rho^x - r^x(\psi' \cos x\theta + \gamma' \sin x\theta)]^n$$

$$= (2\lambda')^n\rho^{nx}.$$ 

If we look at the RHS of the identity, we have

$$(2\lambda')^{n-1}[cTLh(nx) + sTLh(nx)] = (2\lambda')^{n-1}[\lambda' \rho^{nx} + r^{nx}(\psi' \cos nx\theta + \gamma' \sin nx\theta)$$

$$+ \lambda' \rho^{nx} - r^{nx}(\psi' \cos nx\theta + \gamma' \sin nx\theta)]$$

$$= (2\lambda')^{n-1}[2\lambda' \rho^{nx}]$$

$$= (2\lambda')^n\rho^{nx}.$$ 

So the proof is complete.

**Proposition 11** (Sum).

$$2\psi'(cTLh(x + y)) = cTLh(x) \cdot cTLh(y) + sTLh(x) \cdot sTLh(y) + 2\rho^{x+y}(\lambda' \psi' - \lambda^2)$$

$$- 2r^{x+y}\sin x\theta \sin y\theta(\psi'^2 - \gamma'^2)$$

$$2\psi'(sTLh(x + y)) = sTLh(x) \cdot cTLh(y) + cTLh(x) \cdot sTLh(y) + 2\gamma^{x+y}\sin x\theta \sin y\theta(\psi'^2 - \gamma'^2).$$
Proof. Let prove the first identity.

\[ cTLh(x) \cdot cTLh(y) + sTLh(x) \cdot sTh(y) + 2\rho^{x+y}(\alpha \beta - \alpha^2) - 2r^{x+y}\sin x \theta \sin y \theta (\beta^2 - \gamma^2) \]

\[ = [\lambda' \rho^{x} + r^{y}(\psi' \cos x \theta + \gamma' \sin x \theta)][\lambda' \rho^{y} + r^{y}(\psi' \cos y \theta + \gamma' \sin y \theta)] \]

\[ + [\lambda' \rho^{x} - r^{y}(\psi' \cos x \theta + \gamma' \sin x \theta)][\lambda' \rho^{y} - r^{y}(\psi' \cos y \theta + \gamma' \sin y \theta)] \]

\[ + 2\rho^{x+y}(\lambda' \psi' - \lambda^{2}) - 2r^{x+y}\sin x \theta \sin y \theta (\psi'^2 - \gamma'^2) \]

\[ = [2\lambda'^2 \rho^{x+y} + 2r^{x+y}\psi'^2 \cos x \theta \cos y \theta + 2r^{x+y}\gamma'^2 \sin x \theta \sin y \theta \]

\[ + 2r^{x+y}\beta \gamma' \sin x \theta \cos y \theta + 2r^{x+y}\beta \gamma' \cos x \theta \sin y \theta \]

\[ + 2\rho^{x+y}(\lambda' \psi' - \lambda^{2}) - 2r^{x+y}\sin x \theta \sin y \theta (\psi'^2 - \gamma'^2) \]

\[ = 2\psi' [\lambda' \rho^{x+y} + r^{x+y}(\psi' \cos x \theta \cos y \theta - \sin x \theta \sin y \theta) + \gamma'(\sin x \theta \cos y \theta + \cos x \theta \sin y \theta)] \]

\[ = 2\psi' [\lambda' \rho^{x+y} + r^{x+y}(\psi' \cos(x+y) \theta + \gamma' \sin(x+y) \theta)] \]

\[ = 2\psi'(cTLh(x + y)). \]

Now, let prove the second one

\[ sTh(x) \cdot cTh(y) + cTh(x) \cdot sTh(y) + 2\gamma^{x+y} \sin x \theta \sin y \theta (\psi'^2 - \gamma'^2) \]

\[ = [\lambda' \rho^{x} - r^{y}(\psi' \cos x \theta + \gamma' \sin x \theta)][\lambda' \rho^{y} + r^{y}(\psi' \cos y \theta + \gamma' \sin y \theta)] \]

\[ + [\lambda' \rho^{x} + r^{y}(\psi' \cos x \theta + \gamma' \sin x \theta)][\lambda' \rho^{y} - r^{y}(\psi' \cos y \theta + \gamma' \sin y \theta)] \]

\[ + 2\gamma^{x+y} \sin x \theta \sin y \theta (\psi'^2 - \gamma'^2) \]

\[ = [2\lambda'^2 \rho^{x+y} - 2r^{x+y}\psi'^2 \cos x \theta \cos y \theta - 2r^{x+y}\gamma'^2 \sin x \theta \sin y \theta \]

\[ - 2r^{x+y}\beta \gamma' \sin x \theta \cos y \theta - 2r^{x+y}\beta \gamma' \cos x \theta \sin y \theta \]

\[ + 2\gamma^{x+y} \sin x \theta \sin y \theta (\psi'^2 - \gamma'^2) \]

\[ = 2\psi' [\lambda' \rho^{x+y} - r^{x+y}(\psi' \cos x \theta \cos y \theta - \sin x \theta \sin y \theta) + \gamma'(\sin x \theta \cos y \theta + \cos x \theta \sin y \theta)] \]

\[ = 2\psi' [\lambda' \rho^{x+y} - r^{x+y}(\psi' \cos(x+y) \theta + \gamma' \sin(x+y) \theta)] \]

\[ = 2\psi'(sTh(x + y)). \]

\[ \square \]

Corollary 12 (Double argument). By doing $y = x$ in the previous formulas, we have

\[ 2\psi'(cTLh(2x)) = [cTLh(x)]^2 + [sTLh(x)]^2 + 2\rho^{2x}(\lambda' \psi' - \lambda^{2}) - 2r^{2x}\sin^2 x \theta (\psi'^2 - \gamma'^2), \]

\[ 2\psi'(sTLh(2x)) = 2sTLh(x) \cdot cTLh(x) + 2\gamma^{2x} \sin^2 x \theta (\psi'^2 - \gamma'^2). \]

Now we will study the Tribonacci’s properties of the quasi-hyperbolic Tribonacci-Lucas numbers.

Proposition 13 (Recursive relations).

\[ sTLh(x + 1) = sTLh(x) + sTLh(x - 1) + sTLh(x - 2), \]

\[ cTLh(x + 1) = cTLh(x) + cTLh(x - 1) + cTLh(x - 2). \]
Proof. Let us prove the first identity. So let look at the LHS and RHS of that identity, respectively

\[ sTLh(x + 1) \]
\[ = \lambda' \rho^{x+1} - r^{x+1}(\psi' \cos(x+1)\theta + \gamma' \sin(x+1)\theta) \]
\[ = \lambda' \rho^{x+1} - r^{x+1}(\psi'(\cos x \theta \cos \theta - \sin x \theta \sin \theta) + \gamma'(\sin x \theta \cos \theta + \cos x \theta \sin \theta) \]
\[ = \lambda' \rho^{x+1} - r^{x+1}\psi' \cos x \theta \cos \theta + r^{x+1}\psi' \sin x \theta \sin \theta - r^{x+1}\gamma' \sin x \theta \cos \theta - r^{x+1}\gamma' \sin x \theta \cos \theta, \]

\[ sTLh(x) + sTLh(x - 1) + sTLh(x - 2) \]
\[ = [\lambda' \rho^{x} - r^{x}(\psi' \cos x \theta + \gamma' \sin x \theta)] + [\lambda' \rho^{x-1} - r^{x-1}(\psi' \cos(x-1)\theta + \gamma' \sin(x-1)\theta)] \]
\[ + [\lambda' \rho^{x-2} - r^{x-2}(\psi' \cos(x-2)\theta + \gamma' \sin(x-2)\theta)] \]
\[ = \lambda' [\rho^{x} + \rho^{x-1} + \rho^{x-2}] - r^{x} [\psi' \cos x \theta + \gamma' \sin x \theta] \]
\[ - r^{x-1} [\psi'(\cos x \theta \cos \theta + \sin x \theta \sin \theta) + \gamma'(\sin x \theta \cos \theta - \cos x \theta \sin \theta)] \]
\[ - r^{x-2} [\psi'(\cos x \theta \cos 2 \theta + \sin x \theta \sin 2 \theta) + \gamma'(\sin x \theta \cos 2 \theta - \cos x \theta \sin 2 \theta)] \]
\[ = \lambda' [\rho^{x-2}(\rho^{2} + \rho + 1) - r^{x} \psi' \cos x \theta(1 + r^{-1} \cos \theta + r^{-2} \cos 2 \theta) - r^{x} \psi' \sin x \theta(r^{-1} \sin \theta + r^{-2} \sin 2 \theta)] \]
\[ - r^{x} \gamma' \sin x \theta(1 + r^{-1} \cos \theta + r^{-2} \cos 2 \theta) + r^{x} \gamma' \cos x \theta(r^{-1} \sin \theta + r^{-2} \sin 2 \theta) \]
\[ = \lambda' [\rho^{x-1} - r^{x} \psi' \cos x \theta(1 + r^{-1} \cos \theta + r^{-2} \cos 2 \theta) - r^{x} \psi' \sin x \theta(r^{-1} \sin \theta + r^{-2} \sin 2 \theta)] \]
\[ - r^{x} \gamma' \sin x \theta(1 + r^{-1} \cos \theta + r^{-2} \cos 2 \theta) + r^{x} \gamma' \cos x \theta(r^{-1} \sin \theta + r^{-2} \sin 2 \theta). \]

By using \( 1 + r^{-1} \cos \theta + r^{-2} \cos 2 \theta = r \cos \theta \) and \( r^{-1} \sin \theta + r^{-2} \sin 2 \theta = -r \sin \theta \), we obtain the result which we look for.

\[ \square \]

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References


