Application of Chebyshev Polynomials to the Approximate Solution of Singular Integral Equations of the First Kind with Cauchy Kernel on the Real Half-line

J. Ahmadi Shali, A. Jodayree Akbarfam, and M. Kashfi

Abstract In this paper, exact solution of the characteristic equation with Cauchy kernel on the real half-line is presented. Next, the Chebyshev polynomials of the second kind, \( U_n(x) \), and fourth kind, \( W_n(x) \), are used to derive numerical solutions of Cauchy-type singular integral equations of the first kind on the real half-line. The collocation points are chosen as the zeros of the Chebyshev polynomials of the first kind, \( T_n(x) \), and third kind, \( V_{n+1}(x) \). Moreover, estimations of errors of the approximated solutions are presented. The numerical results are given to show the accuracy of the methods presented.

1. Introduction

Let us consider the equation

\[
\frac{1}{\pi} \int_{0}^{+\infty} \frac{\varphi(\sigma)}{\sigma - x} d\sigma + \frac{1}{\pi} \int_{0}^{+\infty} k(x, \sigma) \varphi(\sigma) d\sigma = f(x), \quad x > 0,
\]

where \( k(x, \sigma) \) and \( f(x) \) are given real-valued Hölder continuous functions and \( \varphi(x) \) is an unknown function. The theory of equations of the form (1.1) and their approximate solutions for the case in which the integration line is a closed or open curve of finite length can be found in many references [1, 2, 4, 5, 9].

We apply the transformation of the form (see [7, 8])

\[
\frac{1}{\sigma - x} = \frac{x + 1}{\sigma + 1} \frac{1}{\sigma - x} + \frac{1}{\sigma + 1},
\]

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and rewrite (1.1) in the form
\[
\frac{1}{\pi} \int_{0}^{+\infty} \frac{(x+1)\varphi(\sigma)}{(\sigma+1)(\sigma-x)} d\sigma + \frac{1}{\pi} \int_{0}^{+\infty} \frac{\varphi(\sigma)}{\sigma+1} d\sigma \\
+ \frac{1}{\pi} \int_{0}^{+\infty} k(x, \sigma)\varphi(\sigma) d\sigma = f(x), \quad x > 0.
\] (1.3)

We assume that the behavior of the function \( k(x, \sigma) \) as \( \sigma \to +\infty \) is described by the relation
\[
k(x, \sigma) = k_0(x, \sigma) (\sigma + 1)^{\alpha}, \quad \alpha > 1,
\]
where \( k_0(x, \sigma) \) is a Hölder continuous function. By setting
\[
x = \frac{1+t}{1-t}, \quad \sigma = \frac{1+\tau}{1-\tau},
\] (1.4)
we reduce (1.3) to the form
\[
\frac{1}{\pi} \int_{-1}^{1} \frac{1}{\tau-t} \psi(\tau) d\tau - \frac{1}{\pi} \int_{-1}^{1} \frac{1}{\tau-1} \psi(\tau) d\tau + \frac{1}{\pi} \int_{-1}^{1} k^\ast(t, \tau)\psi(\tau) d\tau = g(t), \quad t \in (-1, 1),
\] (1.5)
where
\[
\psi(\tau) = \varphi\left(\frac{1+\tau}{1-\tau}\right), \quad g(t) = f\left(\frac{1+t}{1-t}\right), \quad k^\ast(t, \tau) = \left(\frac{2}{(1-\tau)^2}\right)k\left(\frac{1+t}{1-t}, \frac{1+\tau}{1-\tau}\right).
\]
We set \( k^\ast(t, \tau) \equiv 0 \) in (1.5) and first analyze the equation
\[
\frac{1}{\pi} \int_{-1}^{1} \frac{1}{\tau-t} \psi(\tau) d\tau - \frac{1}{\pi} \int_{-1}^{1} \frac{1}{\tau-1} \psi(\tau) d\tau = g(t), \quad t \in (-1, 1), \quad g(1) = 0,
\] (1.6)
in two cases.

**Case (I):** If the solution \( \psi(t) \) is sought in the class of Hölder continuous functions on \((-1, 1)\), bounded at the point \( t = 1 \) and unbounded at the point \( t = -1 \), then, in view of [5], we have
\[
\psi(t) = -\frac{1}{\pi} \sqrt{\frac{1-t}{1+t}} \int_{-1}^{1} \sqrt{\frac{1+\tau (g(\tau) + \gamma)}{1-\tau \tau - t}} d\tau,
\] (1.7)
where
\[
\gamma = \frac{1}{\pi} \int_{-1}^{1} \frac{\psi(\tau)}{\tau-1} d\tau.
\] (1.8)
Using the following relation (see [3]):
\[
p.\nu \int_{b}^{x} \left(\frac{b-\tau}{\tau-a}\right)^{\nu} \frac{d\tau}{\tau-\xi} = (\pi \cot \pi \nu)\left(\frac{b-\xi}{\xi-a}\right)^{\nu} - \pi \csc(\pi \nu),
\] (1.9)
we can rewrite (1.7) in the form
\[
\psi(t) = -\sqrt{1 - t} \left( \frac{1}{\pi} \int_{-1}^{1} \sqrt{1 + \tau} \frac{g(\tau)}{\tau - t} d\tau + \gamma \right),
\]
(1.10)
where \( \gamma \) is an arbitrary constant.

The constant \( \gamma \) is uniquely determined if (1.6) is supplemented by the condition
\[
\frac{1}{\pi} \int_{-1}^{1} \frac{\psi(\tau)}{\tau - 1} d\tau = \gamma^*,
\]
(1.11)
equivalent to (1.8). Substituting function (1.10) for \( \psi(t) \) into this relation, we
obtain \( \gamma = \gamma^* \).

**Case (II):** If the solution \( \psi(t) \) is sought in the class of Hölder continuous functions
on \((-1, 1)\), bounded at the points \( t = \pm 1 \), then
\[
\psi(t) = -\sqrt{1 - t^2} \int_{-1}^{1} \frac{(g(\tau) + \gamma)}{\sqrt{1 - \tau^2}} \frac{d\tau}{\tau - t},
\]
(1.12)
provided that
\[
\int_{-1}^{1} \frac{g(\tau) + \gamma}{\sqrt{1 - \tau^2}} d\tau = 0,
\]
(1.13)
(see [5]). Using the following relations
\[
\int_{-1}^{1} \frac{T_n(\tau)}{\sqrt{1 - \tau^2}} \frac{d\tau}{\tau - t} = \begin{cases} \pi U_{n-1}(t), & n > 0, \\ 0, & n = 0, \end{cases}
\]
(1.14)
and
\[
\int_{-1}^{1} \frac{T_0(\tau)}{\sqrt{1 - t^2}} dt = \pi,
\]
(1.15)
we can rewrite (1.12) and (1.13) in the form
\[
\psi(t) = -\frac{1}{\pi} \sqrt{1 - t^2} \int_{-1}^{1} \frac{g(\tau)}{\sqrt{1 - \tau^2}} \frac{d\tau}{\tau - t},
\]
(1.16)
provided that
\[
\gamma = -\frac{1}{\pi} \int_{-1}^{1} \frac{g(\tau)}{\sqrt{1 - \tau^2}} d\tau.
\]
(1.17)
Therefore, in the original variables \( x, \sigma \), the solution of (1.1) where \( k(x, \sigma) \equiv 0 \) is
expressed in the following forms:

**Case (I):** If the solution \( \varphi(x) \) is sought in the class of functions that are Hölder continuous
on \([\epsilon, +\infty), \epsilon > 0\), vanish at infinity, i.e. \( \lim_{x \to \infty} \varphi(x) = 0 \), and can have
an integrable singularity in the neighborhood of \( x = 0 \), then
\[
\varphi(x) = -\frac{1}{\sqrt{x}} \left( \frac{1}{\pi} \int_{0}^{+\infty} \sqrt{\sigma(\sigma + 1) f(\sigma)} \frac{d\sigma}{\sigma - x} \right) d\sigma + \gamma,
\]
(1.18)
where $\gamma$ is an arbitrary constant. Additionally, if the solution $\varphi(x)$ satisfies the condition
\[
-\frac{1}{\pi} \int_0^{+\infty} \frac{\varphi(\sigma)}{\sigma + 1} d\sigma = \gamma^*,
\] (1.19)
where $\gamma^*$ is an arbitrary number, then the unique solution of (1.1) is given by the formula (1.18) with $\gamma = \gamma^*$.

**Case (II):** If the solution $\varphi(x)$ is sought in the class of bounded Hölder functions on $(0, +\infty)$ vanishing at infinity, then
\[
\varphi(x) = -\frac{1}{\pi} \sqrt{x} \int_0^{+\infty} \frac{f(\sigma)}{\sqrt{\sigma}} \frac{d\sigma}{(\sigma - x)},
\] (1.20)
provided that
\[
-\frac{1}{\pi} \int_0^{+\infty} \varphi(\sigma) d\sigma = -\frac{1}{\pi} \int_0^{+\infty} \frac{f(\sigma)}{\sqrt{\sigma}(\sigma + 1)} d\sigma.
\] (1.21)

2. **Approximate Solutions of the Complete Equation**

In this section, we will derive an approximate solution of (1.5) in two cases.

**Case (I):** An approximate solution in the case that the solution of (1.5) is bounded at the point $t = 1$ and unbounded at the point $t = -1$ is expressed of the form
\[
\psi_n(\tau) = \sqrt{\frac{1 - \tau}{1 + \tau}} \sum_{j=0}^{n} \beta_j W_j(\tau),
\] (2.1)
where $W_j$ is the Chebyshev polynomial of the fourth kind which is defined by the following recurrence relation
\[
W_0(\tau) = 1, \quad W_1(\tau) = 2\tau + 1,
\]
\[
W_n(\tau) = 2\tau W_{n-1}(\tau) - W_{n-2}(\tau), \quad n \geq 2.
\] (2.2)

We rewrite (1.5) in the form
\[
\frac{1}{\pi} \int_{-1}^{1} \psi(\tau) d\tau + \frac{1}{\pi} \int_{-1}^{1} k^*(t, \tau)\psi(\tau) d\tau = g(t) + \gamma^*, \quad t \in (-1, 1),
\] (2.3)
where $\gamma^*$ is determined of (1.11). If we substitute (2.1) in (2.3) and use the relation (see [6])
\[
\int_{-1}^{1} \sqrt{\frac{1 - \tau}{1 + \tau}} \frac{W_j(\tau)}{\tau - t} d\tau = -\pi V_j(t),
\] (2.4)
we get
\[
\sum_{j=0}^{n} \beta_j (-V_j(t) + \frac{1}{\pi} Q_j^*(t)) = g(t) + \gamma^*,
\] (2.5)
where
\[ Q_j^*(t) = \int_{-1}^{1} k^*(t, \tau) \sqrt{\frac{1-\tau}{1+\tau}} W_j(\tau) d\tau, \] (2.6)
and \( V_j \) is the Chebyshev polynomial of the third kind which is defined by the following recurrence relation
\[ V_0(\tau) = 1, \quad V_1(\tau) = 2\tau - 1, \]
\[ V_n(\tau) = 2\tau V_{n-1}(\tau) - V_{n-2}(\tau), \quad n \geq 2. \] (2.7)
Using the zeros of \( V_{n+1}(\tau) \),
\[ t_i = \cos \left( \frac{(2i-1)\pi}{2i+3} \right), \quad i = 1, 2, \ldots, n+1, \] (2.8)
as the collocation points, we obtain the coefficients \( \{\beta_j\}_{j=0}^{n} \) by solving the following system of linear equations
\[ \sum_{j=0}^{n} \beta_j \left( -V_j(t_i) + \frac{1}{\pi} Q_j^*(t_i) \right) = g(t_i) + \gamma^*, \quad i = 1, 2, \ldots, n+1. \] (2.9)
In the special case that \( k^*(t, \tau) \equiv 0 \), the approximate solution (1.6) is
\[ \sum_{j=0}^{n} -\beta_j V_j(t_i) = g(t_i) + \gamma^*, \quad i = 1, 2, \ldots, n+1. \] (2.10)

**Case (II):** An approximate solution in the case that the solution of (1.5) is bounded at the points \( t = \pm 1 \) is expressed of the form
\[ \psi_n(\tau) = \sqrt{1 - \tau^2} \sum_{j=0}^{n} \alpha_j U_j(\tau), \] (2.11)
where \( U_j \) is the Chebyshev polynomial of the second kind which is defined by the following recurrence relation
\[ U_0(\tau) = 1, \quad U_1(\tau) = 2\tau, \]
\[ U_n(\tau) = 2\tau U_{n-1}(\tau) - U_{n-2}(\tau), \quad n \geq 2. \] (2.12)
We rewrite (1.5) in the form
\[ \frac{1}{\pi} \int_{-1}^{1} \frac{\psi(\tau)}{\tau-t} d\tau + \frac{1}{\pi} \int_{-1}^{1} k^*(t, \tau) \psi(\tau) d\tau = g(t) + \gamma, \quad t \in (-1, 1), \] (2.13)
where \( \gamma \) is determined of (1.17). If we substitute (2.11) in (2.13) and use the relation (see [6])
\[ \int_{-1}^{1} \frac{\sqrt{1-\tau^2} U_j(\tau)}{\tau-t} d\tau = -\pi T_{j+1}(t), \] (2.14)
then we will obtain
\[ \sum_{j=0}^{n} \alpha_j \left( -T_{j+1}(t) + \frac{1}{\pi} Q_j(t) \right) = g(t) + \gamma, \] (2.15)
where

\[
Q_j(t) = \int_{-1}^{1} k^*(t, \tau) \sqrt{1 - \tau^2} U_j(\tau) d\tau .
\]  

(2.16)

Let \( t_k \) be the zeros of \( T_{n+2}(t) \), i.e.

\[
t_k = \cos \left( \frac{(2k - 1)\pi}{2(n + 4)} \right), \quad k = 1, 2, \ldots, n + 2.
\]  

(2.17)

Substituting the collocation points (2.17) in (2.15), we obtain the coefficients \( \{\alpha_j\}_0^n \) by solving the following system of linear equations

\[
\sum_{j=0}^{n} \alpha_j \left( -T_{j+1}(t_i) + \frac{1}{\pi} Q_j(t_i) \right) = g(t_i) + \gamma, \quad i = 1, 2, \ldots, n + 1.
\]

In the special case that \( k^*(t, \tau) \equiv 0 \), the approximate solution (1.6) is

\[
\sum_{j=0}^{n} -\alpha_j T_{j+1}(t_i) = g(t_i) + \gamma, \quad i = 1, 2, \ldots, n + 1.
\]

3. Error Estimation

Now, we give an error estimation for the approximate solutions of (1.5). Let \( \psi_n(t) \) be approximate solution and \( e_n(t) = \psi_n(t) - \psi(t) \), be the error function associated with \( \psi_n(t) \), where \( \psi(t) \) is the exact solution of (1.5). Since \( \psi_n(t) \) is an approximate solution, it satisfies in

\[
\frac{1}{\pi} \int_{-1}^{1} \frac{\psi_n(\tau)}{\tau - t} d\tau - \frac{1}{\pi} \int_{-1}^{1} \frac{\psi_n(\tau)}{\tau - 1} d\tau + \frac{1}{\pi} \int_{-1}^{1} k^*(t, \tau) \psi_n(\tau) d\tau = g(t) + H_n(t),
\]  

(3.1)

where \( H_n(t) \) is a perturbation term and it is obtained from

\[
H_n(t) = \frac{1}{\pi} \int_{-1}^{1} \frac{\psi_n(\tau)}{\tau - t} d\tau - \frac{1}{\pi} \int_{-1}^{1} \frac{\psi_n(\tau)}{\tau - 1} d\tau + \frac{1}{\pi} \int_{-1}^{1} k^*(t, \tau) \psi_n(\tau) d\tau - g(t).
\]  

(3.2)

Subtracting (1.5) from (3.2), yields the equation

\[
\frac{1}{\pi} \int_{-1}^{1} \frac{e_n(\tau)}{\tau - t} d\tau - \frac{1}{\pi} \int_{-1}^{1} \frac{e_n(\tau)}{\tau - 1} d\tau + \frac{1}{\pi} \int_{-1}^{1} k^*(t, \tau) e_n(\tau) d\tau = H_n(t)
\]  

(3.3)

for the error function \( e_n(t) \). To find an approximation \( \tilde{e}_n(t) \) to \( e_n(t) \), we can solve (3.3) by the same ways as we did for (1.5). In this case, only the function \( g(t) \) will be replaced by the perturbation term \( H_n(t) \). Note that the integrals in above equations are considered as the Cauchy principal value integrals.
4. Numerical Example

In this section, we give a numerical example to clarify accuracy of the presented method. The results of example are reported in Tables 1 and 2. Moreover, we can compare numerical results for $e_n(t) = |\psi_n(t) - \psi(t)|$ and $|\tilde{e}_n(t)|$ in Tables 1 and 2. In the case (I), we consider $\gamma^* = \frac{1}{\pi} \int_{-1}^{1} \frac{g(\tau)}{\sqrt{1-\tau^2}} d\tau$.

Example.

\[
\frac{1}{\pi} \int_{0}^{+\infty} \frac{\varphi(\sigma)}{\sigma - x} d\sigma = \frac{1}{2x+3}, \quad x > 0.
\]  

(4.1)

Table 1. Numerical results in the case (I)

<table>
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<tr>
<th>$x$</th>
<th>$t$</th>
<th>$\psi_n(t)$</th>
<th>$\psi(t)$</th>
<th>$e_n(t)$</th>
<th>$\tilde{e}_n(t)$</th>
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</tr>
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Table 2. Numerical results in the case (II)

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<th>$e_n(t)$</th>
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References


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