Weierstrass Representation for Minimal Surfaces into BCV-Spaces

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Abstract Bianchi-Cartan-Vranceanu spaces (BCV-spaces) are some 3-dimensional homogeneous manifolds equipped with a metric depending on 2 parameters $\kappa$ and $\tau$, and whose isometries groups are of dimension four. In this paper, we describe a Weierstrass-type representation formula for simply connected minimal surfaces immersed into BCV-spaces.

1. Introduction

The topic of Weierstrass representations for minimal surfaces has a long and rich history. It has been extensively investigated since the initial works of Weierstrass [1] and Enneper [2] in the nineteenth century on systems inducing minimal surfaces in $\mathbb{R}^3$. There exist a great number of applications of Weierstrass representations for minimal surfaces in various domains of Mathematics, Physics, Chemistry and Biology [10].

By using the standard harmonic maps equation, Mercuri, Montaldo and Piu gave in [3] a Weierstrass-type representation formula for simply connected minimal surfaces into Riemannian manifolds and they applied the obtained general structure to the case of 3-dimensional Lie groups endowed with left invariant metrics. From this setting, they discussed then some examples of minimal surfaces both in 3-dimensional Heisenberg group $\mathbb{H}_3$ and in $\mathbb{H}^2 \times \mathbb{R}$ where $\mathbb{H}^2$ is the 2-dimensional hyperbolic space.

Let $\kappa$ and $\tau$ be two real numbers and $D_{\kappa,\tau}$ be the domain of $\mathbb{R}^3$ defined by

$$D_{\kappa,\tau} = \left\{ (x, y, z) \in \mathbb{R}^3 \mid 1 + \frac{\kappa}{4}(x^2 + y^2) > 0 \right\}.$$
By considering on \( D_{\kappa, \tau} \) the 2-parameters family of homogeneous Riemannian metrics:

\[
d s^2_{\kappa, \tau} = \frac{d x^2 + d y^2}{(1 + \frac{\kappa}{4}(x^2 + y^2))^2} + \left( d z + \tau \frac{y d x - x d y}{1 + \frac{\kappa}{4}(x^2 + y^2)} \right)^2,
\]

we obtain a 2-parameters family of 3-dimensional Riemannian manifolds \( (D_{\kappa, \tau}, d s^2_{\kappa, \tau}) \), also denoted by \( M^3(\kappa, \tau) \), called Bianchi-Cartan-Vranceanu spaces (BCV-spaces, in short).

The class of BCV-spaces contains all the Riemannian manifolds with 4-dimensional or 6-dimensional isometries groups except the hyperbolic space forms. The BCV-spaces provide model spaces of Thurston’s 3-dimensional geometries (see [12]). In theoretical cosmology, the metrics on BCV-spaces are known as the Bianchi-Kantowski-Sachs type metrics used to construct some homogeneous space-times (see [11]). In these last fifteen years, many differential geometors investigate curves and surfaces with some special properties in BCV-spaces [15, 16]. Surfaces with parallel fundamental forms in BCV-spaces are classified by Belkhelfa, Dillen and Inoguchi in [13], and more generally surfaces with higher order parallel second fundamental forms in BCV-spaces have been classified by J. Van der Veken [14]. In [17] and [18], the authors studied biharmonic curves in BCV-spaces and they obtained interesting classification results. A Weierstrass representation is a description of the surface by some holomorphic functions. D.A. Berdinski and I.A. Taimanov obtained in [9] a Weierstrass type representation for minimal surfaces into BCV-spaces in terms of spinors and Dirac operators.

In this paper, we describe a Weierstrass-type representation formula for minimal surfaces into BCV-spaces in terms of two complex-functions satisfying some integral conditions and we extend thus the results obtained in [3] and [4].

2. Preliminaries

Let \( (M^n, g) \) be an \( n \)-dimensional Riemannian manifold and \( f : \Sigma \subset M \rightarrow M \) be a minimal conformal immersion, where \( \Sigma \) is a Riemann surface. The pull-back bundle \( f^*(TM) \) has a metric and compatible connection, the pull-back connection induced by the Riemannian metric and the Levi-Civita connection of \( M \). Consider the complexified bundle \( \mathcal{E} = f^*(TM) \otimes \mathbb{C} \).

Let \( (u, v) \) be a local coordinates on \( \Sigma \), \( z = u + iv \) the local conformal complex parameter and \( (x_1, \ldots, x_n) \) be a system of local coordinates in a neighborhood \( U \) of \( M \) such that \( U \cap f(\Sigma) \neq \emptyset \). The pull-back connection extends to a complex connection on \( \mathcal{E} \) and it is well known that \( \mathcal{E} \) has a unique holomorphic structure such that a section \( \phi : \Sigma \rightarrow \mathcal{E} \) is holomorphic if and only if

\[
\bar{\nabla}_{\partial_z} \phi = 0,
\]

where \( \bar{\nabla} \) is the pull-back connection on \( \Sigma \).
The induced metric on $\Sigma$ is
\[ ds^2 = \lambda^2 (du^2 + dv^2) = \lambda^2 |dz|^2, \]
and the beltrami-Laplace operator on $\Sigma$, with respect to the induced metric $ds^2$ is given by
\[ \Delta = \lambda^2 \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right). \]

We recall that $f : \Sigma \to M$ is harmonic if and only if its tension field $\tau(f) = \text{trace } \nabla df$ vanishes and for conformal immersions, harmonicity and minimality are equivalent.

Let us consider
\[ \phi = \frac{\partial f}{\partial z} = \frac{1}{2} \left( \frac{\partial f}{\partial u} - i \frac{\partial f}{\partial v} \right). \]

By putting
\[ \phi = \sum_{j=1}^{n} \phi_j \frac{\partial}{\partial x_j} \]
where $\phi_j$ are some complex-valued functions defined on $\Sigma$, the tension field $\tau(f)$ of $f$ can be written as:
\[ \tau(f) = 4\lambda^{-2} \sum_i \left( \frac{\partial \phi_i}{\partial \bar{z}} + \Gamma^i_{jk} \bar{\phi}_j \phi_k \right) \frac{\partial}{\partial x_i} \]
where $\Gamma^i_{jk}$ are the Christoffel symbols of $M$.

The section $\phi$ is then holomorphic if and only if
\[ \nabla_{\bar{z}} \left( \sum_{i=1}^{n} \phi_i \frac{\partial}{\partial x_i} \right) = \sum_{i} \left\{ \frac{\partial \phi_i}{\partial \bar{z}} + \sum_{k,j} \Gamma^i_{jk} \bar{\phi}_j \phi_k \right\} \frac{\partial}{\partial x_i} = 0 ; \]
or equivalently if and only if
\[ \frac{\partial \phi_i}{\partial \bar{z}} + \sum_{k,j} \Gamma^i_{jk} \bar{\phi}_j \phi_k = 0, \quad i = 1, 2, \ldots, n. \tag{2.2} \]

We have then
\[ 4\lambda^{-2} (\nabla_{\bar{z}} \phi) = \tau(f). \]

Thus $f : \Sigma \to M$ is harmonic if and only if $\phi = \frac{\partial f}{\partial z}$ is a holomorphic section of $E$.

Relation (2.2) is a system of first order differential equations in the $\phi_i$, it can be written as:
\[ \frac{\partial \phi_i}{\partial \bar{z}} + 2 \sum_{j > k} \Gamma^i_{jk} \text{Re}(\bar{\phi}_j \phi_k) + \sum_{j} \Gamma^i_{jj} |\phi_j|^2 = 0, \quad i = 1, \ldots, n. \]

This implies that $\frac{\partial \phi_i}{\partial \bar{z}} \in \mathbb{R}$, and ensures that (locally) the 1-forms $\phi_i dz$ do not have real periods as it has been mentioned in [3]. Therefore we have the following:
Proposition 2.1 ([4]). Let \((M,g)\) be a Riemannian manifold and \((x_1,\ldots,x_n)\) local coordinates. Let \(\phi_j, j = 1,\ldots,n\), be complex-valued functions in an open simply connected domain \(\Omega \subset \mathbb{C}\) which are solutions of (2.2). Then the map
\[
f_j(u,v) = 2 \text{Re} \left( \int_{z_0}^z \phi_j dz \right)
\] (2.3)
is well defined and determines a minimal conformal immersion if and only if the following conditions are satisfied:

(i) \(\sum_{j,k=1}^n g_{ij} \phi_j \overline{\phi}_k \neq 0\),

(ii) \(\sum_{j,k=1}^n g_{ij} \phi_j \phi_k = 0\).

In [3], the authors proved that if \(M\) is a Lie group then the system (2.2) has a solution. In the next section we describe a Weierstrass representation for minimal surfaces into 3-dimensional manifold.

3. Weierstrass Representation in 3-dimensional Manifolds

Let \(M^3\) be a 3-dimensional manifold, endowed with an analytic Riemannian metric \(g\). We consider \(M^3\) as a single chart and \((x^1,x^2,x^3)\) a system of coordinates on \(M^3\). By the Gram-Schmidt orthonormalization, we have a basis of vector fields \(E_i, i = 1,2,3\), defined by
\[
E_1 = \frac{1}{A} \left\{ \frac{\partial}{\partial x^1} - \frac{1}{B^2} (g_{12} - g^{33} g_{23} g_{13}) \frac{\partial}{\partial x^2} + g^{33} \left( \frac{1}{B^2} (g_{12} - g^{33} g_{23} g_{13}) g_{23} - g_{13} \right) \frac{\partial}{\partial x^3} \right\},
\]
\[
E_2 = \frac{1}{B} \left( \frac{\partial}{\partial x^2} - g^{33} g_{23} \frac{\partial}{\partial x^3} \right),
\]
\[
E_3 = \frac{1}{\sqrt{g_{33}}} \frac{\partial}{\partial x^3}.
\] (3.1)

where \(g_{ij} = g \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right)\), \(g^{ij} = [g_{ij}]^{-1}\),
\[
A = \sqrt{g_{11} - \frac{1}{B^2} (g_{12} - g^{33} g_{23} g_{13})^2 - g^{33} (g_{13})^2},
\]
\[
B = \sqrt{g_{22} - g^{33} (g_{23})^2}.
\]

In some open set \(\Omega \subset \Sigma\) the section \(\phi = \frac{\partial f}{\partial z} \in \Gamma(f^* (TM) \otimes \mathbb{C})\) can be decomposed with respect to the coordinates vector fields as
\[
\phi = \sum_{i=1}^3 \phi_i \frac{\partial}{\partial x^i} = \sum_{i=1}^3 \psi_i E_i,
\] (3.2)
for some open complex functions $\phi_i, \psi_i : \Omega \to \mathbb{C}$. Moreover, there exists an invertible matrix $\text{Mat} = (m_{ij})_{i,j=1,2,3}$ with the functions entries $m_{ij} : f(\Omega) \to \mathbb{R}$, $i, j = 1, 2, 3$, satisfying

$$\phi_i = \sum_j m_{ij} \psi_j,$$

where

$$\text{Mat} = \begin{bmatrix}
A^{-1} & 0 & 0 \\
-(g_{12} - g^{33} g_{23} g_{13}) (AB^2)^{-1} & B^{-1} & 0 \\
(A)^{-1}[g^{33}((B^2)^{-1} - g^{33} g_{23} g_{13}) g_{23} - g_{13}]] & -g^{33} g_{23} B^{-1} & \sqrt{g^{33}}\end{bmatrix}.$$ (3.3)

By (3.2), we have

$$\mathbb{N}_{\overline{z}} \sum_i \psi_i E_i = \sum_i \left\{ \frac{\partial \psi_i}{\partial \overline{z}} E_i + \sum_{j,k} \psi_k \psi_j g(\nabla E_j, E_i) E_i \right\}.$$ This means that the section $\phi$ is holomorphic if and only if

$$\frac{\partial \psi_i}{\partial \overline{z}} + \sum_{j,k} \psi_k \psi_j g(\nabla E_j, E_i) = 0, \quad i = 1, 2, 3.$$ (3.4)

**Theorem 3.1.** Let $\psi_i, \ i = 1, 2, 3$ be complex-valued functions defined in a simply connected domain $\Omega \subset \mathbb{C}$ such as the following conditions are satisfied:

(i) $\sum_{i=1}^n \psi_i \overline{\psi}_i \neq 0$,

(ii) $\sum_{i=1}^n \psi_i^2 = 0$,

(iii) $\psi_i$ are solutions of (3.4).

Then the map $f := (x^1, x^2, x^3) : \Omega \to (M^3, g)$, defined by

$$x^1(p) = 2 \Re \int_{p_0}^{p} \left( \sqrt{(g_{11} - (g_{33} g_{23})^2)^{-1} (g_{12} - g^{33} g_{13} g_{23})^2 - g^{33} (g_{13})^2} \right)^{-1} \psi_1 d\rho,$$

$$x^2(p) = 2 \Re \int_{p_0}^{p} \left( (g_{22} - g^{33} (g_{23})^2)^{-1} (g_{12} - g_{13} g_{23} g^{33}) \psi_1 \right.\
\left. + \left( \sqrt{g_{22} - g^{33} (g_{23})^2} \right)^{-1} \psi_2 d\rho, \right.$$\n
$$x^3(p) = 2 \Re \int_{p_0}^{p} \left[ (g_{22} - g^{33} (g_{23})^2)^{-1} (g_{23} g^{33} (g_{12} - g_{13} g_{23} g^{33}) - g^{33} g_{13}) \psi_1 \right.\
\left. - \left( \sqrt{g_{22} - g^{33} (g_{23})^2} \right)^{-1} g_{23} g^{33} \psi_2 + \sqrt{g^{33}} \psi_3 d\rho, \right.$$ (3.5)

is a conformal minimal immersion.
Proof. Using (3.1) and (3.2), we get
\[ \phi_1 = \left( \sqrt{g_{11} - g_{22} - g^{33}(g_{23})^2} \right)^{-1} \left( g_{12} - g^{33}g_{23}g_{13} \right)^{-1} \psi_1, \]
\[ \phi_2 = -\left( g_{22} - g^{33}(g_{23})^2 \right)^{-1} \left( g_{12} - g^{33}g_{23}g_{13} \right) \psi_1 + i \left( \sqrt{g_{22} - g^{33}(g_{23})^2} \right)^{-1} \psi_2, \]
\[ \phi_3 = \left( g_{22} - g^{33}(g_{23})^2 \right)^{-1} \left( g_{23}g_{13} - g^{33}g_{12} \right) \psi_1 \]
\[ - \left( \sqrt{g_{22} - g^{33}(g_{23})^2} \right)^{-1} g_{23}g_{13} \psi_2 + 2g^{33} \psi_3. \]

From proposition 2.1, the theorem is proved. \[ \square \]

Remark 3.2. If \( M = \mathbb{R}^3 \) and \( g \) the flat metric on \( M \), we have a Weierstrass representation for minimal surfaces in \( \mathbb{R}^3 \), see [4].

Since the parameter \( z \) is conformal, we have
\[ \psi_1^2 + \psi_2^2 + \psi_3^2 = 0. \tag{3.6} \]

From (3.6) we have
\[ (\psi_1 - i\psi_2)(\psi_1 + i\psi_2) = -\psi_3^2, \]
which suggests the definition of two new complex functions
\[ G := \sqrt{\frac{1}{2}(\psi_1 - i\psi_2)}, \quad H := \sqrt{-\frac{1}{2}(\psi_1 + i\psi_2)}. \tag{3.7} \]

The functions \( G \) and \( H \) are single-valued complex functions which satisfy
\[ \psi_1 = G^2 - H^2, \quad \psi_2 = i(G^2 + H^2), \quad \psi_3 = 2GH. \tag{3.8} \]

In the following, we give a Weierstrass representation for minimal surfaces into BCV-spaces.

4. Weierstrass Representation in BCV-space \( M^3(\kappa, \tau) \)

Let \( \kappa \) and \( \tau \) be two real numbers, with \( \tau \geq 0 \). Bianchi-Cartan-Vranceanu space (BCV-space) \( M^3(\kappa, \tau) \) is defined as the set
\[ D_{\kappa, \tau} = \left\{ (x, y, z) \in \mathbb{R}^3 \middle| 1 + \frac{\kappa}{4}(x^2 + y^2) > 0 \right\} \]
endowed with the metric
\[ ds^2_{\kappa, \tau} = \frac{dx^2 + dy^2}{(1 + \frac{\kappa}{4}(x^2 + y^2))^2} + \left( dz + \tau \frac{ydx - xdy}{1 + \frac{\kappa}{4}(x^2 + y^2)} \right)^2. \tag{4.1} \]

It was Cartan [8] who obtained this family of spaces by classifying of three-dimensional Riemannian manifolds with four-dimensional isometry group. They also appeared in the work L. Bianchi [5, 6], and G. Vranceanu [7]. The complete classification of BCV-spaces is as follows:

- if \( \kappa = \tau = 0 \), then \( M^3(\kappa, \tau) \cong \mathbb{R}^3 \);
- if \( \kappa = 4\tau^2 \neq 0 \), then \( M^3(\kappa, \tau) \cong S^3(\frac{\kappa}{4}) \setminus \{ \infty \} \).
• if $\kappa > 0$ and $\tau = 0$, then $M^3(\kappa, \tau) \cong (S^2(\kappa) \setminus \{\infty\}) \times \mathbb{R}$;
• if $\kappa < 0$ and $\tau = 0$, then $M^3(\kappa, \tau) \cong \mathbb{H}^2(\kappa) \times \mathbb{R}$;
• if $\kappa > 0$ and $\tau \neq 0$, then $M^3(\kappa, \tau) \cong SU(2) \setminus \{\infty\}$;
• if $\kappa < 0$ and $\tau \neq 0$, then $M^3(\kappa, \tau) \cong SL(2, \mathbb{R})$;
• if $\kappa = 0$ and $\tau \neq 0$, then $M^3(\kappa, \tau) \cong \text{Nil}_3$.

By the Gram-Schmidt orthonormalization the following vectors fields form an orthonormal frame on $M^3(\kappa, \tau)$:

$$E_1 = \left(1 + \frac{\kappa}{4}(x^2 + y^2) \frac{\partial}{\partial x} - \tau y \frac{\partial}{\partial z}\right);$$

$$E_2 = \left(1 + \frac{\kappa}{4}(x^2 + y^2) \frac{\partial}{\partial y} + \tau x \frac{\partial}{\partial z}\right); \quad E_3 = \frac{\partial}{\partial z}. \quad (4.2)$$

The corresponding Lie Bracket are

$$[E_1; E_2] = -\frac{\kappa}{2}y E_1 + \frac{\kappa}{2}x E_2 + 2 \tau E_3; \quad [E_1; E_3] = 0; \quad [E_2; E_3] = 0. \quad (4.3)$$

With respect to this orthonormal basis, the Levi-Civita connection can be computed as:

$$\nabla_{E_1} E_1 = \frac{\kappa}{2} y E_2 \quad \nabla_{E_1} E_2 = -\frac{\kappa}{2} y E_1 + \tau E_3, \quad \nabla_{E_1} E_3 = -\tau E_2,$$

$$\nabla_{E_2} E_1 = -\frac{\kappa}{2} x E_2 - \tau E_3, \quad \nabla_{E_2} E_2 = \frac{\kappa}{2} x E_1, \quad \nabla_{E_2} E_3 = \tau E_1,$$

$$\nabla_{E_3} E_1 = -\tau E_2, \quad \nabla_{E_3} E_2 = \tau E_1, \quad \nabla_{E_3} E_3 = 0.$$

We have by Kozul's formula

$$g(\nabla_{E_1} E_1, E_2) = \frac{\kappa}{4} y, \quad g(\nabla_{E_1} E_2, E_1) = -\frac{\kappa}{4} y, \quad g(\nabla_{E_1} E_2, E_3) = \frac{\tau}{2}, \quad g(\nabla_{E_1} E_3, E_2) = -\frac{\tau}{2},$$

$$g(\nabla_{E_2} E_1, E_2) = -\frac{\kappa}{4} x, \quad g(\nabla_{E_2} E_1, E_3) = -\frac{\tau}{2}, \quad g(\nabla_{E_2} E_2, E_1) = \frac{\kappa}{4} x, \quad g(\nabla_{E_2} E_3, E_1) = \frac{\tau}{2},$$

$$g(\nabla_{E_3} E_1, E_2) = -\frac{\tau}{2}, \quad g(\nabla_{E_3} E_2, E_1) = \frac{\tau}{2}.$$

The matrix (3.3) is then given by

$$\begin{bmatrix}
1 + \frac{\kappa}{4}(x^2 + y^2) & 0 & 0 \\
0 & 1 + \frac{\kappa}{4}(x^2 + y^2) & 0 \\
-\tau y & \tau x & 1
\end{bmatrix}.$$

According to (3.4) the section $\phi = \psi_1 E_1 + \psi_2 E_2 + \psi_3 E_3$ is holomorphic if and only if

$$\frac{\partial \psi_1}{\partial \bar{z}} - \frac{\kappa}{4} y \psi_2 \bar{\psi}_1 + \frac{\kappa}{4} x \psi_2 \bar{\psi}_2 + \frac{\tau}{2} \psi_2 \bar{\psi}_3 + \frac{\tau}{2} \psi_3 \bar{\psi}_2 = 0,$$

$$\frac{\partial \psi_2}{\partial \bar{z}} + \frac{\kappa}{4} y \psi_1 \bar{\psi}_2 - \frac{\tau}{2} \psi_1 \bar{\psi}_3 - \frac{\kappa}{4} x \psi_1 \bar{\psi}_2 - \frac{\tau}{2} \psi_3 \bar{\psi}_1 = 0,$$

$$\frac{\partial \psi_3}{\partial \bar{z}} - \frac{\tau}{2} \psi_1 \bar{\psi}_2 + \frac{\tau}{2} \psi_2 \bar{\psi}_1 = 0. \quad (4.4)$$

Let us now write equations (4.4), which ensures that $\phi$ is holomorphic section, in term of the functions $G$ and $H$: 

If $\psi$ satisfies (3.8) then
\[
G \frac{\partial G}{\partial \bar{z}} = \frac{\kappa}{8} y i(|G|^4 - G^2 \bar{H}^2) - \frac{\kappa}{8} x (|G|^4 + G^2 \bar{H}^2) - \frac{i \tau}{2} G \bar{H} (|G|^2 - |H|^2),
\]

\[
H \frac{\partial H}{\partial \bar{z}} = \frac{\kappa}{8} y i(|H|^4 - H^2 \bar{G}^2) + \frac{\kappa}{8} x (|H|^4 + H^2 \bar{G}^2) - \frac{i \tau}{2} H \bar{G} (|G|^2 - |H|^2),
\]

\[
H \frac{\partial G}{\partial \bar{z}} + G \frac{\partial H}{\partial \bar{z}} = -\frac{i \tau}{2} (|G|^4 - |H|^4).
\]

Therefore, Theorem 3.1 can be written as:

**Theorem 4.1.** Let $G$ and $H$ be complex-valued functions defined in a simply connected domain $\Omega \subset \mathbb{C}$ such that:

(i) $G$ and $H$ are not identically zeros.

(ii) $G$ and $H$ are solutions of (4.5)-(4.7).

Then the map $f := (x, y, z) : \Omega \to M^3(\kappa, \tau)$, defined by

\[
x(p) = 2 \text{Re} \int_{p_0}^{p} (1 + \frac{\kappa}{4} (x^2 + y^2)) (G^2 - H^2) d\rho,
\]

\[
y(p) = 2 \text{Re} \int_{p_0}^{p} (1 + \frac{\kappa}{4} (x^2 + y^2)) i (G^2 + H^2) d\rho,
\]

\[
z(p) = 2 \text{Re} \int_{p_0}^{p} -y \tau (G^2 - H^2) + ix \tau (G^2 + H^2) + 2GH d\rho,
\]

is a conformal minimal immersion.

**Proof.** Using (4.2), we get

\[
\phi_1 = \left(1 + \frac{\kappa}{4} (x^2 + y^2)\right) \psi_1, \quad \phi_2 = \left(1 + \frac{\kappa}{4} (x^2 + y^2)\right) \psi_2, \quad \phi_3 = -\tau y \psi_1 + \tau x \psi_2 + \psi_3.
\]

From Theorem 3.1 and (3.8), we have the result. □

**Remark 4.2.** Equations (4.5) and (4.6) are non-linear partial differential equations with non-constant coefficients and it is more complicated to find explicitly solutions $\phi_i$, $i = 1, 2, 3$. By replacing $\kappa$ by $\delta = 1 + \frac{\kappa}{4} (x^2 + y^2)$, with $|\delta| > 2$ and $\delta$ is constant, we obtain the new metric

\[
ds_{\delta, \tau}^2 = \frac{dx^2 + dy^2}{\delta^2} + \left(dz + \tau \frac{y dx - x dy}{\delta}\right)^2,
\]

which looks like a Heisenberg metric but is not isometric to a Heisenberg metric.

By the Gram-Schmidt orthonormalization the following vectors fields form an orthonormal frame on $M^3(\delta, \tau)$:

\[
E_1 = \delta \frac{\partial}{\partial x} - \tau y \frac{\partial}{\partial z}; \quad E_2 = \delta \frac{\partial}{\partial y} + \tau x \frac{\partial}{\partial z}; \quad E_3 = \frac{\partial}{\partial z}.
\]
The corresponding Lie Bracket are
\[
[E_1; E_2] = 2\delta \tau E_3 \quad ; \quad [E_1; E_3] = 0 \quad ; \quad [E_2; E_3] = 0.
\] (4.13)

With respect to this orthonormal basis, the Levi-Civita connection can be computed as:
\[
\nabla_{E_1} E_1 = 0, \quad \nabla_{E_1} E_2 = \delta \tau E_3, \quad \nabla_{E_1} E_3 = -\delta \tau E_2,
\]
\[
\nabla_{E_2} E_1 = -\delta \tau E_3, \quad \nabla_{E_2} E_2 = 0, \quad \nabla_{E_2} E_3 = \delta \tau E_1,
\]
\[
\nabla_{E_3} E_1 = -\delta \tau E_2, \quad \nabla_{E_3} E_2 = \delta \tau E_1, \quad \nabla_{E_3} E_3 = 0.
\]

We have by Kozul's formula
\[
g(\nabla_{E_1} E_2, E_3) = \delta \tau, \quad g(\nabla_{E_1} E_3, E_2) = -\delta \tau, \quad g(\nabla_{E_2} E_1, E_3) = -\delta \tau, \quad g(\nabla_{E_2} E_3, E_1) = \delta \tau.
\]

According to (3.4), the section \( \phi = \psi_1 E_1 + \psi_2 E_2 + \psi_3 \) is holomorphic if and only if
\[
\frac{\partial \psi_1}{\partial \bar{z}} + 2\delta \tau \text{Re}(\psi_2 \bar{\psi}_3) = 0; \quad \frac{\partial \psi_2}{\partial \bar{z}} - 2\delta \tau \text{Re}(\psi_1 \bar{\psi}_3) = 0; \quad \frac{\partial \psi_3}{\partial \bar{z}} - 2i\delta \tau \text{Im}(\psi_1 \bar{\psi}_2) = 0.
\] (4.14)

Equations (4.14) can be written in terms of the functions \( G \) and \( H \) defined by (3.7).
\[
\frac{\partial G}{\partial \bar{z}} = -i\delta \tau \bar{H}(|G|^2 - |H|^2),
\]
\[
\frac{\partial H}{\partial \bar{z}} = -i\delta \tau \bar{G}(|G|^2 - |H|^2).
\] (4.15) (4.16)

Therefore, Theorem 4.1 becomes:

**Theorem 4.3.** Let \( G \) and \( H \) be complex-valued functions defined in a simply connected domain \( \Omega \subset \mathbb{C} \) such that:

(i) \( G \) and \( H \) are not identically zeros.

(ii) \( G \) and \( H \) are solutions of (4.15)-(4.16).

Then the map \( f := (x, y, z): \Omega \to M^3(\delta, \tau) \), defined by
\[
\begin{aligned}
x(p) &= 2 \text{Re} \int_{p_0}^{p} \delta (G^2 - H^2)d\rho, \\
y(p) &= 2 \text{Re} \int_{p_0}^{p} i\delta (G^2 + H^2)d\rho, \\
z(p) &= 2 \text{Re} \int_{p_0}^{p} -\gamma \tau (G^2 - H^2) + ix \tau (G^2 + H^2) + 2GHd\rho,
\end{aligned}
\]
is a conformal minimal immersion.

**References**


