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Research Article

Automorphism Group of Dihedral Groups With Perfect Order Subsets

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Abstract. Let G be a finite group. The set of all possible such orders joint with the number of elements that each order referred to, is called the order classes of G. The order subset of G determined by $x \in G$ is the set of elements in G with the same order as x. A group is said to have perfect order subsets (POS-group) if the cardinality of each order subset divides the group order. In this paper, we compute the order classes of the automorphism group of Dihedral group. Also, we construct a class of POS groups from the automorphism group of the Dihedral group which will serve the solution to the Perfect Order Subset Conjecture.

Keywords. Dihedral group; Order classes; Automorphism; Conjugacy classes

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1. Introduction

The order of an element x in a finite group G is the smallest positive integer k, such that x^k is the group identity and it is denoted by $o(x)$. The set of all available orders of the group G is denoted by $S(G)$, the set of all elements in G of order k is denoted by S_k and the order of this set is $|S_k|$. The idea of perfect order subset of a finite group were introduced, for the first time, by C.E. Finch and L. Jones [8]. They demonstrated several methods for the construction of finite Abelian groups having perfect order subsets and also established a curious connection between such groups and Fermat numbers. In 2003, the same authors [9] considered some of their results

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for finite non-Abelian groups and concluded with the open questions “Are there non-Abelian groups other than S_3 that have a perfect order subsets?” and “If G has perfect order subsets and some odd prime p divides $|G|$, then is it true that $|G|$ is divisible by 3?”. Das [5] considered arbitrary finite groups having perfect order subsets, and obtained some interesting results along with a number of classes of non-Abelian finite groups having perfect order subsets using the idea of semidirect product of finite groups. In [16], it is proved that a finite simple group and a finite group having equal orders and same sets of element orders are isomorphic. In [1] the order classes of dihedral groups are derived. Jones and Toppin [12] discuss certain questions about finite groups G having the property that the cardinalities of all order subsets of G divide the order of G. There are also many studies concerned with determining groups, especially Abelian groups [5, 13–15]. Foote and Reist [10] verified the Perfect Order Subset Conjecture for simple groups for all but one family of finite simple groups. In this paper, we construct a class of perfect order subsets from the automorphism group of the Dihedral group which answers the Perfect Order Subset Conjecture posed by Finch and Jones [8]. Also, we compute the order classes of the automorphism group of Dihedral group D_n.

2. Automorphisms of D_n

Most of the notations, definitions and results we mentioned here are standard and can be found in [2–4, 6, 7, 11]. For any given natural number n denote:

$$\begin{aligned} d(n) &= \text{ the number of positive divisors of } n \\ D(n) &= \text{ the set of all divisors of } n \\ \varphi(n) &= \text{ the number of non- negative integers less than } n \text{ and relatively prime to } n \\ \mathbb{Z}_n &= \text{ the group of integers modulo } n \\ \mathbb{Z}_n^* &= \text{ the group of relatively prime integers modulo } n \end{aligned}$$

Definition 2.1. A prime number of the form $1+2^n$ is called Fermat prime.

Definition 2.2. A group generated by two elements r and s with orders n and 2 such that $srs^{-1} = r^{-1}$ is said to be the nth dihedral group and is denoted by D_n.

Theorem 2.3. *For each divisor d of n, the group $\mathbb{Z}_n$ has exactly $\varphi(d)$ elements of order d, namely $\left\langle \frac{n}{d} \right\rangle$.*

Theorem 2.4. *Let Gbe a group generated by a and b such that $a^n = b^2 = e$ and $bab^{-1} = a^{-1}$. If the size of G is $2n$ then G is isomorphic to D_n.*

By Theorem 2.4, we make an abstract definition for dihedral groups.

Definition 2.5. For $n \ge 3$, let $R_n = \{r_0, r_1, \dots, r_{n-1}\}$ and $S_n = \{s_0, s_1, \dots, s_{n-1}\}$. Define a binary operation on $G_n = R_n \cup S_n$ by the following relations:

$$r_i \cdot r_j = r_{i+j \operatorname{mod}(n)} \qquad r_i \cdot s_j = s_{i+j \operatorname{mod}(n)}$$

$$s_i \cdot s_j = r_{i-j\,\mathrm{mod}(n)} \qquad s_i \cdot r_j = s_{i-j\,\mathrm{mod}(n)} \quad \text{for all } 0 \le i,j \le n-1.$$

Then $(G_n, \cdot)$ is a group of order $2n$.

Note that in the group $(G_n,\cdot)$, the identity element is r_0, $r_i = r_j$ if and only if $i = j\,\mathrm{mod}(n)$, $s_i = s_j$ if and only if $i = j\,\mathrm{mod}(n)$, the inverse of r_i is r_{n-i} and the inverse of s_i is s_i for all $0 \le i$, $j \le n-1$. It is also clear that $r_1^i = r_i$ and $r_j \cdot s_0 = s_j$ for all $0 \le i$, $j \le n-1$. Since G_n is a group of order $2n$ and can be generated by r_1 and s_0 such that:

$$r_1^n = r_n = r_0, s_0^2 = r_0 \text{ and } s_0 r_1 s_0^{-1} = s_0 r_1 s_0 = s_{-1} s_0 = r_{-1} = r_{n-1} = r_1^{-1}.$$

Then by Theorem 2.4, G_n is isomorphic to $D_n =< r_1, s_0 >$. From the group D_n we have the following:

Theorem 2.6. *The number of elements of order* 2 *in* D_n *is*

(i) $n+1$ *if* n *is even, namely* $\{r_{n/2}, s_i : 0 \le i \le n-1\}$.

(ii) n *if* n *is odd, namely* $\{s_i : 0 \le i \le n-1\}$.

Theorem 2.7. *For each divisor* $d(\neq 2)$ *of* n*, the number of elements of order* d *in* D_n *is* $\varphi(d)$ *namely* $\{r_{kn/d} :, 0 \le k \le d-1, (k,d) = 1\}$.

Theorem 2.8. *For* $n \ge 3$*, the number of automorphisms on* D_n *is* $n\varphi(n)$.

Proof. Let $\phi : D_n \to D_n$ be an automorphism. Since $D_n = \langle r_1, s_0 \rangle$, $o(r_1) = n$ and $o(s_0) = 2$, we have

$$D_n = \langle \phi(r_1), \phi(s_0) \rangle, o(\phi(r_1)) = n \text{ and } o(\phi(s_0)) = 2.$$

Since $n \ge 3$, $\phi(r_1) = r_k$ for some $0 \le k \le n-1$ and $(k,n) = 1$. If $\phi(s_0)$ is a rotation, then $D_n \neq \langle \phi(r_1), \phi(s_0) \rangle$ and hence $\phi(s_0) = s_j$ for some $0 \le j \le n-1$. Consequently there are atmost $n\varphi(n)$ automorphism on D_n. That is

$$|\mathrm{Aut}(D_n)| \le n\varphi(n). \tag{2.1}$$

Conversely, for each $0 \le k$, $j \le n-1$ and $(k,n) = 1$ define a map $\phi_{k,j} : D_n \to D_n$ by

$$\phi_{k,j}(r_i) = r_{ik\,\mathrm{mod}(n)} \text{ and } \phi_{k,j}(s_i) = s_{ki+j\,\mathrm{mod}(n)} \quad \text{for all } 0 \le i \le n-1.$$

Now, we will prove $\phi_{k,j}$ is an automorphism on D_n. For this, let $0 \le i$, $t \le n-1$. Then

$$\begin{aligned}
\phi_{k,j}(r_i r_t) &= \phi_{k,j}(r_{i+t}) \\
&= r_{(i+t)k\,\mathrm{mod}(n)} \\
&= r_{ik\,\mathrm{mod}(n)} r_{tk\,\mathrm{mod}(n)} \\
&= \phi_{k,j}(r_i)\phi_{k,j}(r_t),
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
\phi_{k,j}(s_i s_t) &= \phi_{k,j}(r_{i-t}) \\
&= r_{(i-t)k\,\mathrm{mod}(n)} \\
&= s_{ik+j\,\mathrm{mod}(n)} s_{tk+j\,\mathrm{mod}(n)} \\
&= \phi_{k,j}(s_i)\phi_{k,j}(s_t),
\end{aligned} \tag{2.3}$$

$$\phi_{k,j}(r_i s_t) = \phi_{k,j}(s_{i+t})$$

$$
\begin{aligned}
&= r_{(i+t)k+j\,\mathrm{mod}(n)} \\
&= r_{ik\,\mathrm{mod}(n)} s_{tk+j\,\mathrm{mod}(n)} \\
&= \phi_{k,j}(r_i)\phi_{k,j}(s_t)
\end{aligned}
\tag{2.4}
$$

and

$$
\begin{aligned}
\phi_{k,j}(s_t r_i) &= \phi_{k,j}(s_{t-i}) \\
&= s_{(t-i)k+j\,\mathrm{mod}(n)} \\
&= s_{tk+j\,\mathrm{mod}(n)} r_{ik\,\mathrm{mod}(n)} \\
&= \phi_{k,j}(s_t)\phi_{k,j}(r_i)
\end{aligned}
\tag{2.5}
$$

From (2.2), (2.3), (2.4) and (2.5), $\phi_{k,j} : D_n \to D_n$ is a homomorphism. Since $\phi_{k,j}(r_1) = r_k$, $\phi_{k,j}(s_0) = s_j$ and $(k,n) = 1$, we have

$$
\begin{aligned}
& D_n = \langle r_k, s_j \rangle \subseteq \phi_{k,j}(D_n) \subseteq D_n \\
\implies\ & \phi_{k,j}(D_n) = D_n \\
\implies\ & \phi_{k,j} \text{ is onto.}
\end{aligned}
$$

Since D_n is a finite group and $\phi_{k,j}$ onto, $\phi_{k,j}$ is one-one. Hence $\phi_{k,j} : D_n \to D_n$ is an automorphism for all $0 \le k, j \le n-1$ and $(k,n) = 1$. Therefore

$$
|\mathrm{Aut}(D_n)| \ge n\varphi(n). \tag{2.6}
$$

From (2.1) and (2.6), we have

$$
|\mathrm{Aut}(D_n)| = n\varphi(n).
$$

□

Corollary 2.9. $\mathrm{Aut}(D_n) = \{\phi_{k,j} : 0 \le k, j \le n-1, (k,n) = 1\}$, *where* $\phi_{k,j}$ *is the unique automorphism on* D_n *induced by the map* $r_1 \to r_k$ *and* $s_0 \to s_j$.

For a natural number n, define

$$
\overline{G}_n = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a \in \mathbb{Z}_n^*, b \in \mathbb{Z}_n \right\}.
$$

Then $\overline{G}_n$ is a group of order $n\varphi(n)$ with respect to matrix multiplication. The identity element of $\overline{G}_n$ is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and the inverse of $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} a^{-1} & -ba^{-1} \\ 0 & 1 \end{bmatrix}$.

Theorem 2.10. $\mathrm{Aut}(D_n)$ *is isomorphic to* $\overline{G}_n$ *for all* n.

Proof. Define $\psi : \mathrm{Aut}(D_n) \to \overline{G}_n$ by

$$
\psi(\phi_{i,j}) = \begin{bmatrix} i & j \\ 0 & 1 \end{bmatrix}, \quad 0 \le i, j \le n-1, (i,n) = 1. \tag{2.7}
$$

Now,

$$
\begin{aligned}
\phi_{i,j} \circ \phi_{k,l}(r_1) &= \phi_{i,j}(r_k) \\
&= r_{ik\,\mathrm{mod}(n)} \\
&= \phi_{ik\,mod(n), li+j\,\mathrm{mod}(n)}(r_1)
\end{aligned}
$$

and

$$\begin{aligned}\phi_{i,j}\circ\phi_{k,l}(s_0) &= \phi_{i,j}(s_l)\\ &= s_{li+j\,\mathrm{mod}(n)}\\ &= \phi_{ik\,\mathrm{mod}(n),li+j\,\mathrm{mod}(n)}(s_0).\end{aligned}$$

Therefore

$$\phi_{i,j}\circ\phi_{k,l} = \phi_{m,t}, \quad \text{where } m = ik\,\mathrm{mod}(n) \text{ and } t = li+j\,\mathrm{mod}(n).$$

So,

$$\begin{aligned}\psi(\phi_{i,j}\circ\phi_{k,l}) &= \psi(\phi_{m,t})\\ &= \begin{bmatrix} m & t\\ 0 & 1\end{bmatrix}\\ &= \begin{bmatrix} ik\,\mathrm{mod}(n) & (li+j)\,\mathrm{mod}(n)\\ 0 & 1\end{bmatrix}\\ &= \begin{bmatrix} i & j\\ 0 & 1\end{bmatrix}\begin{bmatrix} k & l\\ 0 & 1\end{bmatrix}\\ &= \psi(\phi_{i,j})\psi(\phi_{k,l}).\end{aligned}$$

Hence ψ is a homomorphism from $\mathrm{Aut}(D_n)$ onto $\overline{G}_n$. Assume

$$\begin{aligned} & \psi(\phi_{i,j}) = \psi(\phi_{k,l})\\ \Longrightarrow\quad & \begin{bmatrix} i & j\\ 0 & 1\end{bmatrix} = \begin{bmatrix} k & l\\ 0 & 1\end{bmatrix}\\ \Longrightarrow\quad & i=k \text{ and } j=l\\ \Longrightarrow\quad & \phi_{i,j} = \phi_{k,l}\\ \Longrightarrow\quad & \psi \text{ is one-one.}\end{aligned}$$

Clearly ϕ is onto also. Therefore ϕ is an isomorphism from $\mathrm{Aut}(D_n)$ to $\overline{G}_n$. □

3. Order Classes of the Automorphism Group of Dihedral Groups

In this section, we compute the order class of the automorphism group of D_n and using this we characterize its POS property.

Definition 3.1. Let G be a finite group and $S(G) = \{o(x) : x \in G\}$. For each $k \in S(G)$, denote $S_k = \{x \in G : o(x) = k\}$. Then the order class of G is defined as the set

$$\{(k, |S_k|) : k \in S(G)\}.$$

A group G is said to have perfect order subsets (in short, G is called a POS-group) if the number of elements in each S_k is a divisor of $|G|$.

Theorem 3.2. *For $n > 1$, $\varphi(n)$ divides n if and only if $n = 2^k 3^l$, where $k \ge 1$ and $l \ge 0$.*

Proof. Suppose $\varphi(n)$ divides n. We will show that $n = 2^k 3^l$, where $k \ge 1$ and $l \ge 0$.
Let $n = p_1^{n_1} p_2^{n_2} \dots p_t^{n_t}$ be the prime factorization of n, where p_i's are distinct primes, $n_i \ge 1$ for

all $1 \le i \le t$ and $p_1 < p_2 < \ldots < p_t$. Then

$$\begin{aligned}\varphi(n) &= n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_t}\right)\\ &= p_1^{n_1-1}p_2^{n_2-1}\ldots p_t^{n_t-1}(p_1-1)(p_2-1)\ldots(p_t-1).\end{aligned}$$

Therefore $\varphi(n)$ divides n if and only if $(p_1-1)(p_2-1)\ldots(p_t-1)$ divide $p_1p_2\ldots p_t$. Since $p_1p_2\ldots p_t$ is a square free, $L=(p_1-1)(p_2-1)\ldots(p_t-1)$ is also square free. If $t \ge 3$, then L is divisible by 4, which is impossible, so $t \le 2$. If $p_1 > 2$, then p_1-1 is even and hence $p_1p_2\ldots p_t$ is even, impossible. Therefore $p_1 = 2$ and $k \ge 1$. If $t > 1$, since p_2-1 and p_2 are relatively prime, p_2-1 divides 2 and $p_2 = 3$. Hence if $n > 1$ and $\varphi(n)$ divides n, then n has the form 2^k3^l with $k \ge 1$ and $l \ge 0$.

Conversely, suppose that $n = 2^k3^l$ with $k \ge 1$ and $l \ge 0$. If $l = 0$, then $\varphi(n) = 2^{k-1}$ which divides n. If $l \ge 0$, then $\varphi(n) = 2^k3^{l-1}$ which divides n. Hence for $n > 1$, $\varphi(n)$ divides n if and only if $n = 2^k3^l$, where $k \ge 1$ and $l \ge 0$. □

Corollary 3.3. *For $n > 1$, $\varphi(k)$ divides n for all $k \in D(n)$ if and only if $n = 2^k3^l$, where $k \ge 1$ and $l \ge 0$.*

The proof is clear from the fact that if d divides n implies $\varphi(d)$ divides $\varphi(n)$.

By Theorems 2.6 and 2.7, we have the following:

Theorem 3.4. *The order class of D_n is*

(i) $\{(d,\varphi(d)),(2,n) : d \in D(n)\}$ *if n is odd.*

(ii) $\{(d,\varphi(d)),(2,n+1) : d \in D(n), d \ne 2\}$ *if n is even.*

Theorem 3.5. *D_n is a POS group if and only if $n = 3^k$ for some $k \ge 1$.*

Theorem 3.6. *Let p be a prime number. Then*

$$1+z+z^2+\ldots+z^{k-1} \equiv 0 \bmod(p)$$

for all $z \in \mathbb{Z}_p^*$, $z \ne 1$ and $o(z) = k$ in $\mathbb{Z}_p^*$.

Proof. Since $o(z) = k$ in $\mathbb{Z}_p^*$, $z^k \equiv 1 \bmod(p)$. Now,

$$\begin{aligned}(1+z+z^2+\ldots+z^{k-1})(z-1) &= z^k-1\\ &\equiv 0 \bmod(p).\end{aligned} \tag{3.1}$$

Since $1 < z < p$, we have $z-1$ is not congruent to $0 \bmod(p)$. Hence by (3.1),

$$1+z+z^2+\ldots+z^{k-1} \equiv 0 \bmod(p)$$

for all $z \in \mathbb{Z}_p^*$, $z \ne 1$ and $o(z) = k$ in $\mathbb{Z}_p^*$. □

Theorem 3.7. *Let p be a prime number. Then the order class of* Aut(D_p) *is*

$$\{(1,1),(p,p-1),(k,p\varphi(k)) : k \in D(p-1), k \ne 1\}.$$

Proof. By Theorem 2.10, $\mathrm{Aut}(D_n)$ is isomorphic to $\overline{G}_n$.
Let $y \in \mathbb{Z}_p$. Then for any $m \in \mathbb{N}$,

$$\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}^m = \begin{bmatrix} 1 & my \\ 0 & 1 \end{bmatrix}$$

$$\implies \quad o\left(\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}\right) \text{ in } \overline{G}_p = o(y) \text{ in } \mathbb{Z}_p \tag{3.2}$$

Let $x \in \mathbb{Z}_p^*$, $x \neq 1$ and $y \in \mathbb{Z}_p$. Then for any $m \in \mathbb{N}$,

$$\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}^m = \begin{bmatrix} x^m & (1+x+\ldots+x^{m-1})y \\ 0 & 1 \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}^m = I$$

$$\implies \quad x^m \equiv 1 \bmod (p)$$

$$\implies \quad m \geq o(x) \text{ in } \mathbb{Z}_p^*. \tag{3.3}$$

Let $o(x) = k$ in $\mathbb{Z}_p^*$. Then

$$\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} x^k & (1+x+\ldots+x^{k-1})y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{(by Theorem 3.6)}$$

$$\implies \quad o\left(\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}\right) \leq k = o(x) \text{ in } \mathbb{Z}_p^* \tag{3.4}$$

From (3.3) and (3.4), we get

$$o\left(\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}\right) = o(x) \text{ in } \mathbb{Z}_p^* \tag{3.5}$$

for all $x(\neq 1) \in \mathbb{Z}_p^*$ and $y \in \mathbb{Z}_p$. From (3.2) and (3.5)

$$S(\overline{G}_p) = S(\mathbb{Z}_p^*) \cup S(\mathbb{Z}_p) = \{p, k : k \in D(p-1)\}.$$

Also, $|S_1| = 1$, $|S_p| = p-1$ and $S_k = p\varphi(k)$ for all $k \in D(p-1)$ and $k \neq 1$. Hence the order class of $\mathrm{Aut}(D_p)$ is

$$\{(1,1), (p, p-1), (k, p\varphi(k)) : k \in D(p-1), k \neq 1\}.$$

□

Corollary 3.8. $\mathrm{Aut}(D_p)$ *is a POS group if and only if* $p = 1 + 2^k 3^l$ *for some* $k \geq 1$ *and* $l \geq 0$.

Proof. We have $|\mathrm{Aut}(D_p)| = p(p-1)$. Hence by the above theorem, $\mathrm{Aut}(D_p)$ is a POS group if and only if $\varphi(k)$ divide $p-1$ for all $k \in D(p-1)$. Hence by Corollary 3.3 $\mathrm{Aut}(D_p)$ is a POS group if and only if $n = 1 + 2^k 3^l$ for some $k \geq 1$ and $l \geq 0$. □

Theorem 3.9. *Let* p *be a prime number of the form* $1 + 2^k (k \geq 2)$. *Then* $\mathrm{Aut}(D_p)$ *is a non-Abelian POS group whose order is not divisible by* 3.

Proof. Since p is a prime of the form $1 + 2^k (k \geq 2)$, by Corollary 3.9, $\mathrm{Aut}(D_p)$ is a POS group. Again, $|\mathrm{Aut}(D_p)| = p \times 2^k$. Since $p \geq 5$, $|\mathrm{Aut}(D_p)|$ is not divisible by 3.

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$ be two elements in $\overline{G}_p = \mathrm{Aut}(D_p)$. Then

$$AB = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}.$$

Hence $AB \neq BA$ and $\mathrm{Aut}(D_p)$ is non-Abelian group. □

Corollary 3.10. *For all Fermat prime p,* $\mathrm{Aut}(D_p)$ *is a non-Abelian POS group whose order is not divisible* 3.

We conclude the section by providing answers to the two open questions posed by Finch and Jones [8].

Question 1. Are there non-Abelian groups other than S_3 that have a perfect order subsets?

Question 2. If G has perfect order subsets and some odd prime p divides $|G|$, then is it true that $|G|$ is divisible by 3?

Many authors furnished different examples and counter examples for each of these conjectures, but by Corollary 3.10, for all Fermat prime p, $\mathrm{Aut}(D_p)$ is a non-Abelian POS group whose order is not divisible 3 gives a family of groups that simultaneously answers both the questions.

4. Conclusion

In this paper, we computed the order class $\{(1,1),(p,p-1),(k,p\varphi(k)) : k \in D(p-1), k \neq 1\}$ of $\mathrm{Aut}(D_p)$ for every prime number p. Also, we proved that for all Fermat prime p, $\mathrm{Aut}(D_p)$ is a non-abelian POS group whose order is not divisible 3.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] B. Al-Hasanat, A. Ahmad and H. Sulaiman, Order classes of dihedral groups, *AIP Conference Proceedings* **1605**(1) (2014), 551 – 556, DOI: 10.1063/1.4887648.

[2] S.R. Cavior, The subgroups of the dihedral group, *Mathematics Magazine* **48**(2) (1975), p. 107, DOI: 10.1080/0025570X.1975.11976454.

[3] K. Conrad, *Dihedral groups II*, URL: `https://kconrad.math.uconn.edu/blurbs/grouptheory/dihedral2.pdf` (accessed: December 2020).

[4] K. Conrad, *Dihedral groups*, URL: `http://www.math.uconn.edu/~kconrad/blurbs/grouptheory/dihedral.pdf` (accessed: December 2020).

[5] A.K. Das, On finite groups having perfect order subsets, *International Journal of Algebra* **3**(29) (2009), 629 – 637.

[6] X. Du and W. Shi, Finite groups with conjugacy classes number one greater than its same order classes number, *Communications in Algebra* **34** (2006), 1345 – 1359, DOI: 10.1080/00927870500454638.

[7] D.S. Dummit and R.M. Foote, *Abstract Algebra*, Wiley Hoboken (2003).

[8] C.E. Finch and L. Jones, A curious connection between Fermat numbers and finite groups, *American Mathematical Monthly* **109**(6) (2002), 517 – 524, DOI: 10.1080/00029890.2002.11919881.

[9] C.E. Finch and L. Jones, Nonabelian groups with perfect order subsets, *JP Journal of Algebra, Number Theory and Application* **3**(1) (2003), 13 – 26.

[10] R.M. Foote and B.M. Reist, The perfect order subset conjecture for simple groups, *Journal of Algebra* **391** (2013), 1 – 21, DOI: 10.1016/j.jalgebra.2013.05.029.

[11] J.A. Gallian, *Contemporary Abstract Algebra*, D.C. Heath and Company (1994).

[12] L. Jones and K. Toppin, On three questions concerning groups with perfect order subsets, *Involve, A Journal of Mathematics* **4**(3) (2012), 251 – 261, DOI: 10.2140/involve.2011.4.251.

[13] S. Libera and P. Tlucek, Some perfect order subset groups, *Pi Mu Epsilon Journal* **11**(9) (2003), 495 – 498, https://www.jstor.org/stable/24340521.

[14] R. Shen, A note on finite groups having perfect order subsets, *International Journal of Algebra* **4**(13-16) (2010), 643 – 646.

[15] N.T. Tuan and B.X. Hai, On perfect order subsets in finite groups, *International Journal of Algebra* **4**(21-24) (2010), 1021 – 1029.

[16] A.V. Vasil'ev, M.A. Grechkoseeva and V.D. Mazuro, Characterization of the finite simple groups by spectrum and order, *Algebra and Logic* **48**(6) (2009), 385 – 409, https://link.springer.com/content/pdf/10.1007/s10469-009-9074-9.