Some Results on Strong Edge Geodetic Problem in Graphs

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Abstract. For a graph $G(V(G), E(G))$, the problem to find a $S \subseteq V(G)$ where every edge of the graph $G$ is covered by a unique fixed geodesic between the pair of vertices in $S$ is called the strong edge geodetic problem and the cardinality of the smallest such $S$ is the strong edge geodetic number of $G$. In this paper the strong edge geodetic problem for product graphs are studied and also some results for general graphs are derived.

Keywords. Strong edge geodetic number; Strong geodetic number; Edge geodetic number; Geodetic set

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1. Introduction

Consider a graph $G(V(G), E(G))$, with order $|V(G)|$ and size $|E(G)|$. An $(x-y)$ geodesic is the length of the shortest path between the vertices $x$ and $y$. For a graph $G$, the length of the maximum geodesic is called the graph diameter, denoted as $\text{diam}(G)$. Harary et al. introduced a graph theoretical parameter in [2] called the geodetic number of a graph and it was further studied in [1]. In [2] the geodetic number of a graph is defined as follows, let $I[u,v]$ be the set of all vertices lying on some $u-v$ geodesic of $G$, and for some non-empty subset $S$ of $V(G)$, $I[S] = \bigcup_{u,v \in S} I[u,v]$. The set $S$ of vertices of $G$ is called a geodetic set of $G$, if $I[S] = V$. A geodetic set of minimum cardinality is called minimum geodetic set of $G$. The cardinality of the minimum
Theorem 1. For a graph $G$ is the geodetic number $g(G)$ of $G$. The geodetic set decision problem is NP-complete [12]. The set $S \subseteq V(G)$ is an edge geodetic cover of $G$ if every edge of $G$ is contained in the geodesics between some pair of vertices in $S$, and the cardinality of minimum edge geodetic cover is called the edge geodetic number of $G$ denoted as $g_1(G)$ [13]. Strong geodetic problem is a variation of geodetic problem and is defined in [11] as follows. For a graph $G(V(G), E(G))$, given a set $S \subseteq V(G)$, for each pair of vertices $(x, y) \subseteq S$, $x \neq y$, let $\bar{g}(x, y)$ be a selected fixed shortest path between $x$ and $y$. Let $\bar{I}(S) = \{\bar{g}(x, y) : x, y \in S\}$ and $V(\bar{I}(S)) = \bigcup_{\bar{P} \in \bar{I}(S)} V(\bar{P})$. If $V(\bar{I}(S)) = V$ for some $\bar{I}(S)$, then $S$ is called a strong geodetic set. The cardinality of the minimum strong geodetic set is the strong geodetic number of $G$ and is denoted by $sg(G)$. The edge version of the strong geodetic problem is defined in [10] i.e. for a graph $G(V(G), E(G))$, a set $S \subseteq V(G)$ is called a strong edge geodetic set if for any pair $x, y \in S$ a shortest path $P_{xy}$ can be assigned such that $\cup_{(x,y) \in (S)} E(P_{xy}) = E(G)$. The cardinality of the smallest strong edge geodetic set of $G$ is called the strong edge geodetic number and is denoted as $sg_e(G)$.

2. Strong Edge Geodetic Number of Graphs

**Theorem 1.** For a graph $G(V(G), E(G))$ with diameter $d \geq 2$, the strong edge geodetic number, $sg_e(G) \geq \left\lceil \frac{d + \sqrt{d^2 + 8d|E|}}{2d} \right\rceil$.

**Proof.** Consider a graph $G(V(G), E(G))$ with diameter $d \geq 2$ and assume that the set $S \subseteq V(G)$ forms a minimum strong edge geodetic set for $G$. Then the edges of $G$ are covered with $\binom{|S|}{2}$ geodesics and each geodesic covers at most $d$ edges. Thus $|E| \leq \binom{|S|}{2} d$. Considering only the positive and the integer roots, the inequality reduces to $|S| \geq \left\lceil \frac{d + \sqrt{d^2 + 8d|E|}}{2d} \right\rceil$. □

The above bound attains equality when $G \cong P_n$ $(n \geq 2)$.

**Theorem 2.** For a graph $G$ with order $n \geq 2$, $sg_e(G) = 2$ if and only if $G \cong P_n$.

**Proof.** For $P_n = v_0v_1\ldots v_n$, the geodesic between the vertices $v_0$ and $v_n$ covers all the edges of $P_n$ and thus $sg_e(P_n) = 2$. Conversely let $G$ be a graph with $sg_e(G) = 2$ and let $S = \{u, v\}$ be the minimum strong edge geodetic set of $G$. Then all the edges of $G$ lies on the $u – v$ geodesic. Thus $G \cong P_n$. □

A graph is geodetic if for every pair for vertices the shortest path between them is unique. An undirected graph $G = (V, E)$ is said to be geodetic, if between any pair of vertices $x, y \in V(G)$ there is a unique shortest path [5].

**Theorem 3.** For a connected non-complete geodetic graph $G$ on $n$ vertices with minimum cutset of independent vertices $C \subset V(G)$, $sg_e(G) \leq n - c$ where $|C| = c$.

**Proof.** Consider a connected non-complete geodetic graph $G$ with minimum cutset of independent vertices $C = \{c_1, c_2, \ldots, c_v\}$, such that the $V(G) \setminus C$ splits $G$ into at least two components. It is clear that every vertex in $C$ is adjacent to at least one vertex in each component of $G \setminus C$. Let $S = V(G) \setminus C$. Consider an edge $xy \in E(G)$. This edge $xy$ can be of two forms.
Theorem 6. For a split graph $G$. Then $xy$ itself forms a unique fixed geodesic joining $x$ and $y$, where $(x,y) \in S$.

Case ii: Suppose $xy$ is an edge such that $y \in C$ and $x \in S$. Since $G$ is a geodetic graph, there exists a vertex $z \in G \setminus C$ such that the edge $xy$ lies on a unique fixed geodesic $xyz$. It is clear that the vertices $x$ and $z$ lies on two different components of $G \setminus C$. Thus $S$ forms a strong edge geodetic set of $G$ and therefore $sg_e(G) \leq n - c$.

Theorem 5 ([13]). For a graph $G$ of order $n$ and exactly one vertex of degree $(n - 1)$, $g_1(G) = n - 1$.

Proof. For any connected graph $G$, $g_1(G) \leq sg_e(G)$ and from Theorem 4, $sg_e(G) \geq n - 1$. Let $v \in V(G)$ be the unique vertex of degree $n - 1$. Let $vx \in E(G)$. This implies that there exists at least one vertex $x'$ where $x$ and $x'$ are not adjacent. Let $Q = V(G) \setminus \{v\}$. Clearly, the edge $vx$ lies on the unique fixed geodesic $xvx'$. Also, any edge of $G$ which are not incident with $v$ lies on a fixed geodesic between the vertices of $Q$. Thus $sg_e(G) \leq n - 1$. Hence for a graph $G$ of order $n$ and exactly one vertex of degree $(n - 1)$, $sg_e(G) = n - 1$.

The converse of the above theorem need not be true. For $G = C_4$, $sg_e(G) = 3$, but $G$ does not have any vertex of degree 3.

Corollary 1. For a graph $G$ of order $n \geq 3$ with a cut vertex of degree $n - 1$, $sg_e(G) = n - 1$.

Proof. Consider a connected graph $G$ of order $n \geq 3$ with a cut vertex $x$ of degree $n - 1$. This implies that $x$ is the only vertex of degree $n - 1$. Then it follows from Theorem 5 that $sg_e(G) = n - 1$.

Theorem 6. For a split graph $G(V, E)$ with complete set $K$, maximum stable set $T$, $sg_e(G) \geq \frac{2|E| + s_1^2 + 2s_2^2 + 3s_1 + 2s_1s_2}{2(2s_1 + s_2)}$, where $s_1 = |T|$ and $s_2$ denotes number of simplicial vertices in complete set $K$.

Proof. Let $S$ be a minimum strong edge geodetic set of $G$ and let $A$ be set of simplicial vertices in $K$ with $|A| = s_2$. Also, let $B$ be set of non simplicial vertices in $S$, $|B| = s_3$. Clearly, $T \subseteq S$, $A \subseteq S$ and $|S| = s_1 + s_2 + s_3$.

The strong edge geodetic set contains three components including the stable set, the set of simplicial vertices in $K$ and the set of non-simplicial vertices. It is clear that $s_1$ and $s_2$ are fixed, hence to get the minimum strong edge geodetic set, $s_3$ should be minimized. The geodesics between any pair of vertices in the stable set $T$ cover a maximum of three edges. Also, the geodesics between any pair of vertices with one vertex in $T$ and other vertex in $A$ cover at most two edges. Thus the optimization problem reduces to

\[
\text{minimize}(s_3)
\]

subject to: $0 \leq (s_1 + s_2 + s_3) \leq n$,

\[
|E| \leq 3 \left(\frac{s_1}{2}\right) + 2s_2s_1 + 2s_3s_1 + s_3s_2.
\]
From this we obtain \( S \geq \frac{2|E|+s_1^2+2s_2^2+3s_1+2s_1s_2}{2(2s_1+s_2)} \).

Therefore, \( sg_e(G) \geq \frac{2|E|+s_1^2+2s_2^2+3s_1+2s_1s_2}{2(2s_1+s_2)} \). \hfill \square

**Corollary 2.** Suppose \( A = \phi \). Then \( sg_e(G) \geq \frac{2|E|+s_1(s_1+3)}{4s_1} \).

**Theorem 7 ([13]).** For any graph \( G \) with at least two vertices of degree \( n-1 \), \( g_1(G) = n \).

**Theorem 8.** If \( G \) is a graph with order \( n \) and at least two vertices of degree \( n-1 \), then \( sg_e(G) = n \).

**Proof.** For any graph \( G \), \( g_1(G) \leq sg_e(G) \) and from Theorem 7 For \( G \) with order \( n \) and at least two vertices of degree \( n-1 \), \( sg_e(G) = n \). \hfill \square

**Corollary 3.** For a complete graph \( K_n \) with order \( n \geq 4 \) and edge \( e_1 \in E(G) \), \( sg_e(K_n - e_1) = n \).

**Corollary 4.** For a complete graph \( K_n \) with order \( n \geq 6 \) and edges \( \{e_1,e_2\} \in E(G) \), \( sg_e(K_n - \{e_1,e_2\}) = n \).

**Result 1.** Let \( G \) be a cactus graph with \( m \) simplicial vertices and \( r \) cycles then, \( sg_e(G) \leq m + 2r \). Also, the bound is sharp (refer Figure 2.1).

![Cactus graph with \( sg_e(G) = 4 \)](image)

**Figure 2.1.** Cactus graph with \( sg_e(G) = 4 \)

**Result 2.** Let \( G \) be a block cactus graph with \( m \) simplicial vertices and \( r \) cycles then, \( sg_e(G) \leq m + 2r \). The bound is sharp for the graph (refer Figure 2.2).

![Block cactus graph with \( sg_e(G) = 9 \)](image)

**Figure 2.2.** Block cactus graph with \( sg_e(G) = 9 \)
Let $G$ and $H$ be two graphs of order $n$ and $m$, respectively. The corona product $G \circ H$ is defined as the graph obtained from $G$ and $H$ by taking one copy of $G$ and $n$ copies of $H$ and joining by an edge each vertex from the $i^{th}$-copy of $H$ with the $i^{th}$-vertex of $G$. The order of $G \circ H$ is $o(G) o(H) + o(G)$ [12].

**Theorem 9.** For graphs $G$ and $H$ with $o(G) \geq 2$, $sg_e(G \circ H) = o(G) o(H)$.

**Proof.** Consider connected graphs $G$ and $H$ with $o(G) = n$ and $o(H) = m$. The corona product of graphs $G$ and $H$ i.e. $(G \circ H)$ contains $n$ copies of $H$. Let $S$ be the set that contains all vertices in each copies of $H_i$ in $G \circ H$. Since $G$ and $H$ are assumed to be connected, there exists a fixed geodesics between the vertices in each $H_i$ that covers all the edges of $H_i$. The unique fixed geodesics between the vertices of $H_i$ and $H_j$, $i \neq j$ covers the edges of $G$ and the edges connecting $H_i$ and $G$. Thus $sg_e(G \circ H) \leq o(G) o(H)$.

Suppose there exists a set $S'$ such that $o(S') < o(G) o(H)$ is a strong edge geodetic set of $(G \circ H)$. Then there exists $H_j$ such that $v \in H_j (1 \leq j \leq n)$ and $v \in S'$. Clearly, $e = uv$ where $u \in G$ is not covered by any $u - v$ geodesic of $S'$. Therefore $sg_e(G \circ H) = o(G) o(H)$. \hfill \qed

**Definition 1 ([16]).** Let $G$ be a connected graph. An ordered set $S = \{u_1, u_2, \ldots, u_k\}$ of vertices of $G$ is a linear geodetic set of $G$ if for each $x$ in $G$, there exists an index $i$, $1 \leq i < k$ such that $x$ lies on $u_i - u_{i+1}$ geodesic in $G$ and the minimum cardinality of the linear geodetic set is the linear geodetic number, $g_l(G)$.

**Definition 2.** Consider a graph $G$ with an ordered set of vertices, $S = \{x_1, x_2, \ldots, x_k\}$. Then $S$ is a strong linear geodetic if for each $x \in V \setminus S$, there exists an index $i$, $1 \leq i < k$ such that $x$ lies on the unique fixed geodesic between $x_i$ and $x_{i+1}$. The minimum cardinality of a strong linear geodetic set is called the strong linear geodetic number, denoted as $sg_l(G)$.

![Figure 2.3](image_url)  

**Figure 2.3.** Graph $G$ with linear geodetic set $\{v_1, v_2\}$ and strong linear geodetic set $\{v_1, v_2, v_3, v_4\}$

In [8] an upper bound of strong geodetic problem for Cartesian product of two graphs is given as follows: If $G$ and $H$ are graphs, then $sg(G \square H) \leq \min\{sg(H) o(G) - sg(H) + 1, sg(G) o(H) - sg(G) + 1\}$. Later in [3], the bound is improved and is given as: If $G$ and $H$ are graphs, then $sg(G \square H) \leq \min\{sg(H) o(G) - o(H) + 1, sg(G) o(H) - o(G) + 1\}$. We give a new bound for $sg(G \square H)$ using strong linear geodetic numbers of $G$ and $H$.  

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Theorem 10. For connected graphs $G$ and $H$,
$$sg(G \square H) \leq \min \left\{ sg_i(H) + (o(G) - 1) \left[ \frac{sg_i(H)}{2} \right], sg_i(G) + (o(H) - 1) \left[ \frac{sg_i(G)}{2} \right] \right\}.$$ 

Proof. Let $o(G) = n$, $o(H) = m$ and $V(G \square H) = \{(u_i,v_j) | u_i \in V(G), v_j \in V(H)\}$, where $(i = 1,2,\ldots,n)$ and $(j = 1,2,\ldots,m)$. Let $m$ layers of $G$ in $G \square H$ be: $G^1,G^2 \ldots G^m$. Let $P \subseteq V(G)$ where $P = \{g_1,g_2,\ldots,g_k\}$ be the strong linear geodetic set of $G$. Define a set $S = \{(g_i,h) | i = 1,2,\ldots,k, h \in V(H)\}$. Clearly, $\tilde{I}(S) = V$. Let $v \in V(H)$. Define $S' = \{(g_i,v)i = 1,2,\ldots,k\} \cup \{(g_j,h), h \in V(H)\}$, where $j = 2,4,\ldots$ $k$ if $k \equiv 0 \mod 2$ or $j = 2,4,\ldots(k-1)$ if $k \equiv 1 \mod 2$. Let $(g,h) \in V(G \square H)$. If $h = v$ then $(g,h) \in \tilde{I}(S_v)$ where $S_v \subseteq S$ and $S_v = \{(g_i,v) | i = 1,2,\ldots,k\}$. Now, if $h \neq v$ then there exists vertices $g_i$ and $g_{i+1}$ such that $(g,h) \in \tilde{I}((g_i,h),(g_{i+1},h))$. Without loss of generality assume $i+1$ is odd. Consider the shortest path $(g_i,h) \to (g_{i+1},h) \to (g_{i+1},v)$. Then it is straightforward to see that $(g_i,h) \in \tilde{I}((g_i,h),(g_{i+1},v))$. This implies $I(S) = I(S')$. Thus $S'$ forms a strong geodetic for $G \square H$. Therefore, $sg(G \square H) \leq sg_i(G) + (o(H) - 1) \left[ \frac{sg_i(G)}{2} \right]$.

By symmetry, $sg(G \square H) \leq sg_i(H) + (o(G) - 1) \left[ \frac{sg_i(H)}{2} \right]$. Thus for connected graphs $G$ and $H$, $sg(G \square H) \leq \min \{sg_i(H) + (o(G) - 1) \left[ \frac{sg_i(H)}{2} \right], sg_i(G) + (o(H) - 1) \left[ \frac{sg_i(G)}{2} \right] \}$. \hfill \Box

Definition 3 (Strong shortest path union cover, [18]). Let $G(V,E)$ be a connected graph. For a fixed vertex $v \in V(G)$, let $I_{u,v}$ be the selected fixed shortest path between $u$ and $v$ denoted by $\tilde{P}_{u,v}$. A set $S \subseteq V(G)$ is a strong shortest path union cover of $G$ if for all $e \in E(G)$ there exists $u \in S$ such that $e$ lies on a fixed $\tilde{P}_{u,v}$ for some $v \in V(G)$. The cardinality of the minimum strong shortest path union cover set is the strong shortest path union cover number denoted by $SSP_a(C(G))$.

Theorem 11. For connected graphs $G$ and $H$, $sg_e(G \square H) \leq \min \{SSP_a(C(H)o(G) + (o(H) - SSP_a(C(G))(sg_e(G) - 1)), SSP_a(C(G)o(H) + (o(G) - SSP_a(C(G))(sg_e(G) - 1))\}.$

Proof. Let $o(G) = n$, $o(H) = t$. Then $V(G \square H) = \{(g,h) | g \in V(G), h \in V(H)\}$. Let $t$ layers of $G$ in $G \square H$ be: $G^1,G^2 \ldots G^t$. Let $T \subseteq V(H)$ as $T = \{h_1,h_2,\ldots,h_k\}$ be a minimum strong shortest path union cover set of $H$ and and $T' = V(H)\setminus T$. Also, let $A = \{g_1,g_2,\ldots,g_m\}$ be a minimum strong edge geodetic set for $G$ and consider $B = \{g_1,g_2,\ldots,g_{m-1}\}$. Now, we will prove that $S = \{(g,h_1) | g \in V(G), h_1 \in T\} \cup \{(g,j,h) | g \in A, h \in T'\}$ forms a strong edge geodetic cover for $G \square H$. Let $e \in E(G \square H)$. Suppose $e = (gh)(gh')$ of $G \square H$. Since $T$ is a strong shortest path union cover set for $H$, there exists $h_i \in S$ such that the edge $hh'$ lies on the selected fixed shortest path $\tilde{P}_{h,v}$ for some $v \in V(H)$. Consider the vertex $(g,v)$ in $(G')$ layer and let $S_v \subseteq S$ where $S_v = \{(g_j,v), j = 1,2,\ldots,m\}$ is a strong edge geodetic set for the $G^v$ layer. But it is also a strong geodetic set. Therefore, there exists $(g_r,v),(g'_r,v) \in S_v$ such that the vertex $(g,v)$ lies on the selected fixed geodesic between $(g_r,v)$ and $(g'_r,v)$. Now clearly, $e = (gh)(gh')$ lies on the fixed geodesic $(g,v) \to (g',v)$. Suppose $e$ lies on some $G_k$ of $G \square H$. Clearly, $S_k \subseteq S$ where $S_k = \{(g_i,v_1)i = 1,\ldots,m,v_1 \in V(H)\}$ is a strong geodetic set for $G_k$ layer. Then there exist vertices $(g_r,v_k)$ and $(g_s,v_k)$ where $r,s = 1,2,\ldots,m$ such that $e$ lies on a unique fixed shortest path between them. Suppose $S_k = \{(g_i,v_k) | g \in V(G), v_k \in T\}$, then $e \in \tilde{I}(S_k)$. Therefore $S$ is strong edge geodetic set for $G \square H$. Consider, $S' = \{(g,h_i) | g \in V(G), h_i \in T\} \cup \{(g,j,h) | g,j \in B, h \in T'\}$.
The graphs in Figure 2.4 and Figure 2.5 are strong linear edge geodetic graphs. This implies that for a strong linear edge geodetic set $S_g$ is called the strong linear edge geodetic number, denoted as $sg_e$ such that $g_1, u, g_2, u, \ldots, (g_m, u)$ is a strong geodetic set in $G$, there exists an index $1 \leq k \leq m - 1$. It is straightforward to see that $e'$ lies on the fixed geodesic $(g_m, h_j) \rightarrow (g_m, u) \rightarrow (g_k, u)$. Suppose $e' \in (g_s, h), 1 \leq s \leq m - 1$ geodesic in $G^h$ layer. Since $h \in V(H)$, there exits an edge $hx, x \in V(H)$. Also, since $T$ is a strong shortest path union cover set of $H$, there exits $h_t \in T$, $1 \leq t \leq k$ and $y \in V(H)$ such that $hy \in \tilde{P}_{h,t}$. Clearly, $(g_t, h_t) \in S'$ and $e'$ lies on the fixed geodesic $(g_m, h_t) \rightarrow (g_m, h) \rightarrow (g_s, h)$. Thus $\tilde{I}(S) = \tilde{I}(S')$ and $S'$ is strong edge geodetic set for $G^h$. This implies that $sg_e(G^h) \leq SSP_u C(H) o(G) + (o(H) - SSP_u C(H))(sg_e(G) - 1)$. By symmetry, $sg_e(G^h) \leq SSP_u C(G) o(H) + (o(G) - SSP_u C(G))(sg_e(G) - 1)$. Thus for connected graphs $G$ and $H$, $sg_e(G^h) \leq \min(\text{SSP}_u C(H) o(G) + (o(H) - SSP_u C(H))(sg_e(G) - 1), \text{SSP}_u C(G) o(H) + (o(G) - SSP_u C(G))(sg_e(G) - 1))$. □

**Remark 1.** In the above theorem the strong shortest path union cover number cannot be replaced by the shortest path union cover number.

**Definition 4 ([15]).** Let $G$ be a connected graph. An ordered set $S = \{u_1, u_2, \ldots, u_k\}$ of vertices of $G$ is a linear edge geodetic set of $G$ if for each edge $e = xy$ in $G$, there exists an index $i$, $1 \leq i < k$ such that $e$ lies on $u_i - u_{i+1}$ geodesic in $G$ and the minimum cardinality of the linear edge geodetic set is the linear geodetic number.

**Definition 5.** Consider a graph $G$ with an ordered set of vertices of $G$, $S = \{x_1, x_2, \ldots, x_k\}$. Then $S$ is a strong linear edge geodetic if for every edge $e \in \text{E}(G)$, there exists an index $i$, $1 \leq i < k$ such that $e$ lies on unique fixed $x_i - x_{i+1}$ geodesic. The minimum cardinality of a strong linear edge geodetic set is called the strong linear edge geodetic number, denoted as $sg_{ls}(G)$.

A graph $G$ is called strong linear edge geodetic graph if $G$ has a strong linear edge geodetic set. The graphs in Figure 2.4 and Figure 2.5 are strong linear edge geodetic graphs.

**Figure 2.4.** Strong linear edge geodetic graph

**Figure 2.5.** Strong linear edge geodetic graph
A graph may have linear edge geodetic set but not strong linear edge geodetic set (refer Figure 2.6).

**Theorem 12.** For connected strong linear edge geodetic graphs $G$ and $H$,

$$sg_e(G \Box H) \leq \min \left\{ SSP_u C(H)o(G) + (o(H) - SSP_u C(G)) \left[ \frac{sg_e(G)}{2} \right], \right.$$  

$$SSP_u C(G)o(H) + (o(G) - SSP_u C(G)) \left[ \frac{sg_e(H)}{2} \right] \right\}.$$  

**Proof.** Let $o(G) = n$, $o(H) = m$. Then $V(G \Box H) = \{(g,h) \in V(G), h \in V(H)\}$. Let $m$ layers of $G$ in $G \Box H$ be: $G^1, G^2, \ldots, G^m$. Let $T \subset V(H)$ as $T = \{h_1, h_2, \ldots, h_k\}$ be a minimum strong shortest path union cover set of $H$ and and $T^c = V(H) \setminus T$. Also, let $A = \{g_1, g_2, \ldots, g_k\}$ be a minimum strong linear edge geodetic set for $G$ and consider a set $B = \{g_2, g_4, \ldots, g_j \}$ where $j = m$ if $m \equiv 0 \mod 2$ or $j = (m - 1)$ if $m \equiv 1 \mod 2$. The set $S = \{(g_i, h_i) \in V(G), h_i \in T\} \cup \{(g_j, h) \in A, h \in T^c\}$ forms a strong edge geodetic set for $G \Box H$ (by first part of Theorem 11). Let $S' = \{(g_i, h_i) \in V(G), h_i \in T\} \cup \{(g_j, h) \in B, h \in T^c\}$. The proof of $I(S) = I(S')$ is similar to that of Theorem 10. Therefore, we omit the proof. \hfill \Box

**Theorem 13.** For complete graphs $K_m$ and $K_n$, where $m \geq n \geq 2$, $sg_e(K_m \Box K_n) = mn - n$.

**Proof.** For any connected graph $G$, $g_1(G) \leq sg_e(G)$. From [14], $g_1(K_m \Box K_n) = mn - n$. Thus $sg_e(G) \geq mn - n$.

Define $n$ layers of $K_m$ in $K_m \Box K_n$ as follows: $K^1_m, K^2_m, \ldots, K^n_m$. Let $S = V(K_m) = \{g_1, g_2, \ldots, g_m\}$ and $T = V(K_n) = \{h_1, h_2, \ldots, h_n\}$. Define the set $T = S \times T \setminus \{(g_i, h_i)\}$ where $i = 1, 2, \ldots, n$. We prove that $T$ forms a strong edge geodetic set for $K_m \Box K_n$. The set $T$ contains $(m - 1)$ vertices from each $n$ layers of $K_m$ in $K_m \Box K_n$. The geodesics between these $(m - 1)$ vertices in $K^i_m$, $1 \leq i \leq n$ will leave the edges having one endpoint at $(g_i, h_i)$ uncovered. But the diameter of $K_m \Box K_n$ is equal to 2. Thus these uncovered edges can be covered by the unique fixed geodesic between $(g_i, h_i) \rightarrow (g_i, h_k)$ where $i \neq k$. Hence $T$ forms a strong edge geodetic set for $K_m \Box K_n$. This implies that $sg_e(G) \leq mn - n$. Thus for complete graphs $K_m$ and $K_n$ where $m \geq n \geq 2$, $sg_e(K_m \Box K_n) = mn - n$. \hfill \Box

Consider $K_{x} \Box P_m$. Let $K^1_{x}, K^2_{x}, \ldots, K^m_{x}$ be the $m$ layers of $K_n$. Let $S$ be the strong edge geodetic set of $K_{x} \Box P_m$. Observe that each geodesic between the vertices of $S$ can cover at most one edge of $K^i_{x}$, $1 \leq i \leq m$. Suppose $(k, h), (k, h') \in S$, then the geodesic between $(k, h)$ and $(k, h')$ where $k \in K_n$, $h, h' \in P_m$ will not cover any edge of $K^i_{x}$, $1 \leq i \leq m$. If $|S| = nr$, $1 < r < m$,
then the geodesic between the vertices of $S$ can cover at most $\binom{r}{2} - n\binom{r}{2}$ edges of $K^i_n$, $1 \leq i \leq m$. Therefore $m\binom{r}{2} \leq (r^n) - n\binom{r}{2}$. On simplification, $m \leq r^2$. By taking $S$ as the vertices of $r$-layers of $K^i_n (1, 4, 9, \ldots, r^2)$ layers of $K^n$, it is straightforward to see that $S$ is a strong edge geodetic set of $K^n\Box P^r_2$. Therefore, we have the following theorem:

**Theorem 14.** For $K^n\Box P^r_2$, $sg_e(K^n\Box P^r_2) = nr$, $r \geq 2$.

**Corollary 5.** If $r^2 \leq m \leq (r + 1)^2$, then $nr \leq sg_e(K^n\Box P^m) \leq n(r + 1)$.

**Theorem 15.** For two positive integers $a, b$ where $a \leq b \leq 2a - 3$ and $a \geq 2$, there exists a connected graph $G$ with $sg(G) = a$ and $sg_e(G) = b$.

**Proof.** For $G = K^1_1, sg = a = sg_e$. If $a < b \leq 2a - 3$, let $G$ be a graph obtained from $P^4: yv_1xu_a$. Add new vertices $u_1, u_2, \ldots, u_{a-1}$ and join each $u_i$ to the vertex $y$. Also, add vertices $v_i$ and $w_j$ where $2 \leq i \leq 2a - b - 1$ and $1 \leq j \leq b - a$. Now in $G$ each $v_i$ and $w_j$ is joined to both $y$ and $x$. Also, the vertices $u_1, u_2, \ldots, w_{b-a}$ induces a complete graph on $b-a$ vertices (refer Figure 2.7). Thus, the graph $G$ is obtained. It is clear that $S = \{u_1, u_2, \ldots, u_{a-1}, u_a\}$ forms a strong geodetic set for $G$. But $S$ does not form a strong edge geodetic set for $G$. It can be easily seen that $T = S \cup \{w_1, w_2, \ldots, w_{b-a}\}$ forms a strong edge geodetic set for $G$. Thus, $sg = a$ and $sg_e = b - a + a - 1 + 1 = b$. $\square$

![Figure 2.7. Graph G with sg(G) = a and sg_e(G) = b](image-url)

The strong geodetic problem was first studied by Manuel et al. in [11] and it is proved to be NP-complete for general graphs. Later, in [7], the computational complexity for strong geodetic problem is given for bipartite and multipartite graphs where it is proved to be NP-complete for both bipartite and multipartite graphs. We prove that the strong geodetic problem is NP-complete for chordal graphs. The proof of NP-completeness of strong geodetic problem for chordal graphs is by polynomial reduction from dominating set problem.
Theorem 16. Strong geodetic problem for chordal graphs is NP-complete.

Proof. Given a chordal graph $G(V, E)$, construct $\bar{G}(\bar{V}, \bar{E})$ as follows: $\bar{V} = V \cup V' \cup V'' \cup \{a\}$ where $V'$ is a set of independent vertices and $V''$ induces another set of independent vertices disjoint from $V'$. The edge set of $\bar{G}$ is $\bar{E} = E \cup E' \cup E''$ where $E' = \{uu' \mid u \in V, u' \in V'\} \cup \{u'u'' \mid u' \in V', u'' \in V''\}$ and $E'' = \{ua \mid u \in V\} \cup \{u'a \mid u' \in V'\}$. Clearly $\bar{G}$ is a chordal graph.

Since the vertices of $V''$ forms a set of simplicial vertices and they are the elements of any strong geodetic set in $\bar{G}$. Let $D$ be any domination set of $G$. We will prove that $D \cup V''$ forms a strong geodetic set for $\bar{G}$. Consider the geodesics between a vertex $y'' \in V''$ and $x \in D$. If $x'' = y''$, then there exists unique geodesic $y'' - x$. Suppose $y \in N(x)$, then there can be two different $y'' - x$ geodesics each of length 3. The $y'' - x$ geodesics can be $y''y'yx$ or $y''y'ax$. Clearly, these geodesics cover all vertices of $\bar{G}$ in a unique fixed path. Thus $D \cup V''$ forms a Strong geodetic set for $\bar{G}$.

Conversely, assume that $D \cup V''$ is a strong geodetic set of $\bar{G}$. Let $D' = D \setminus V'$, where $D' \subset V(G)$. Consider $u' \in V' \cap D$ and $x \in D'$. It is straightforward to see that, the vertices covered by the geodesic $u' - x$ can be covered by the $(u'' - x)$ geodesic. Therefore, the vertices of $V' \cap D$ are redundant. This implies that $D' \cup V''$ is a strong geodetic set for $\bar{G}$. Clearly, $D'$ is a dominating set for $G$.

3. Conclusion

This paper contains some general bounds for strong edge geodetic problem, strong edge geodetic problem for specific graph classes and bounds for corona and Cartesian product of graphs. In future, we hope to study the strong edge geodetic problem for other graph products like lexicographic product, strong product and rooted product. Further, the strong edge geodetic problem for some networks can also be found.

Competing Interests
The authors declare that they have no competing interests.

Authors’ Contributions
All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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