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Research Article

On the Solution of the Delay Differential Equation via Laplace Transform

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Abstract. In this paper, we consider the initial-value problem for a linear second order delay differential equation. We use Laplace transform method for solving this problem. Furthermore, we present examples provided support the theoretical results.

Keywords. Delay differential equation; Initial-value problem; Laplace transform method

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1. Introduction

Delay Differential Equations (DDEs) appear in many fields such as engineering science, physics, biosciences, economics [2,7,10,14,15,18,24,27,31]. For example, neural networks [20], population dynamics [17], time lag cell growth [1], bistable device [30] are modelled these equations. In recent years, there are many researchers who have investigated oscillation, Hopf bifurcation, numerical aspect and stability analysis for DDEs [3, 4, 6, 8, 16, 23, 29, 34]. On the other hand, Laplace transform is widely used to solve application problems in mathematics [11, 26], physics [13, 33], economics [9]. Therefore, it is a useful tool not only for mathematicians but also for physicists and engineers.

Motivated by the above works, we investigate the following delay differential problem:

$$u''(t)+au'(t)+bu'(t-r)+cu(t)+du(t-r)=f(t), \quad t>0, \tag{1.1}$$

$$u(t)=\varphi(t),\ -r\le t\le 0; \quad u'(0)=\gamma, \tag{1.2}$$

where a, b, c, d are real constants, $f(t)$ and $\varphi(t)$ are given real valued and sufficiently smooth functions, γ is a real number and r is a positive constant large delay. Furthermore, the existence and uniqueness of solution to DDEs is discussed in [5, 10, 12, 19, 21, 22, 32].

It is the aim of this work to develop the Laplace transform method to establish the exact solution a class of second order delay differential equation.

This paper is organized as follows. In Section 2, we give some definitions and preliminaries that we use in the next sections. In Section 3, we present the main results including the solution of the problem (1.1)-(1.2) with Laplace transformation method. In Section 4, we present two examples to illustrate the results.

2. Definitions and Preliminaries

Definition 2.1 ([28])**.** Suppose that g is a real-valued function of the variable $t > 0$ and s is a real parameter. The Laplace transform is defined by

$$\mathscr{L}\{g(t)\} = \int_0^\infty e^{-st} g(t)dt. \tag{2.1}$$

Theorem 2.2 ([28])**.** *Let g be a real function that has the following properties:*

(1) *g is piecewise continuous in every finite interval $0 < t < t_1$ $(t_1 > 0)$.*

(2) *g is of exponential order; that is, there exists $(\alpha$, $M > 0$, and $t_0 > 0$ such that*

$$e^{-\alpha t}|g(t)| < M \quad \text{for } t > t_0.$$

Then the Laplace transform

$$\int_0^\infty e^{-st} g(t)dt$$

of g exist for $s > \alpha$.

Theorem 2.3 ([28])**.** *Suppose that $g(t), g'(t), \dots, g^{(n-1)}(t)$ real functions are continuous on $(0,\infty)$ and of exponential order α, while $g^{(n)}(t)$ is piecewise continuous on $[0,\infty)$. Then*

$$\mathscr{L}\{g^{(n)}(t)\} = s^n \mathscr{L}\{g(t)\} - s^{n-1}g(0) - s^{n-2}g'(0) - \dots - g^{(n-1)}(0). \tag{2.2}$$

Theorem 2.4 (Lerch's theorem, [28])**.** *Distinct continuous functions on $[0,\infty)$ have distinct Laplace transforms.*

It means that if we restrict our attention to functions that are continuous on $[0,\infty)$, then the inverse transform

$$\mathscr{L}^{-1}\{G(s)\} = g(t)$$

is uniquely defined.

Theorem 2.5 (Gronwall's inequality, [25])**.** *Let $g(t)$, $K(t) \geq 0$, $h(t) \geq 0$ real functions are continuous on $(0,\infty)$. If*

$$v(t) \leq g(t) + K(t)\int_0^t h(\tau)v(\tau)d\tau,$$

then

$$v(t) \leq g(t) + K(t)\int_0^t g(\tau)h(\tau)e^{\int_\tau^t h(\xi)K(\xi)d\xi}d\tau.$$

3. Main Results

In this section, we use the Laplace transform method to solve the problem (1.1)-(1.2).

Theorem 3.1. *Let $f(t)$ in (1.1) satisfies the conditions in Theorem 2.2. Then, the Laplace transformation of $u(t)$ (which is the exact solution of (1.1)-(1.2)) and $u'(t)$ and $u''(t)$ (it's first and second order derivatives) exist for all s provided $s > \alpha$.*

Proof. Firstly, integrating the relation (1.1) over $(0,t)$, we get

$$u'(t)-\gamma+a[u(t)-\varphi(0)]+b[u(t-r)-\varphi(-r)]+c\int_0^t u(x)dx+d\int_0^t u(x-r)]dx=\int_0^t f(x)dx. \tag{3.1}$$

If we integrate this equation again over $(0,t)$, we have

$$\begin{aligned} u(t)-\varphi(0)-[\gamma+a\varphi(0)+b\varphi(-r)]t+\int_0^t[a+c(t-x)]u(x)dx+\int_0^t[b+d(t-x)]u(x-r)]dx \\ =\int_0^t (t-x)f(x)dx. \end{aligned} \tag{3.2}$$

Meanwhile, replacing integral variable by $x=\tau+r$, we can write

$$\begin{aligned} \int_0^t u(x-r)dx &= \int_{-r}^{t-r} u(\tau)d\tau \\ &= \int_{-r}^{0}\varphi(\tau)d\tau+\int_0^{t-r} u(\tau)d\tau \end{aligned}$$

and

$$\begin{aligned} \int_0^t (t-x)u(x-r)dx &= \int_{-r}^{t-r}(t-\tau-r)u(\tau)d\tau \\ &= \int_{-r}^{0}(t-\tau-r)\varphi(\tau)d\tau+\int_0^{t-r}(t-\tau-r)u(\tau)d\tau. \end{aligned}$$

If we consider these expressions in Eq. (3.2), we can write

$$u(t)+h_1(t)+\int_0^t[a+c(t-x)]u(x)dx+\int_0^{t-r}[b+d(t-x-r)]u(x)]dx=\int_0^t(t-x)f(x)dx, \tag{3.3}$$

where

$$h_1(t)=-\varphi(0)-[\gamma+a\varphi(0)+b\varphi(-r)]t+\int_{-r}^{0}[b+d(t-\tau-r)]\varphi(\tau)d\tau.$$

Then,

$$\left|\int_0^{t-r}[b+d(t-x-r)]u(x)]dx\right| \le \int_0^t |[b+d(t-x-r)]u(x)]|\,dx$$

when this inequality is taken into account in (3.3), we can write

$$\begin{aligned} |u(t)| &\le |h_1(t)|+\left|\int_0^t[a+c(t-x)]u(x)dx\right|+\left|\int_0^{t-r}[b+d(t-x-r)]u(x)]dx\right|+\left|\int_0^t(t-x)f(x)dx\right| \\ &\le |h_1(t)|+\int_0^t|a+c(t-x)|\,|u(x)|\,dx+\int_0^t|b+d(t-x-r)|\,|u(x)|\,dx+\int_0^t(t-x)\,|f(x)|\,dx \\ &\le |h_1(t)|+\int_0^t[|a+c(t-x)|+|b+d(t-x-r)|]\,|u(x)|\,dx+\int_0^t(t-x)\,|f(x)|\,dx \\ &\le |h_1(t)|+h_2(t)\int_0^t|u(x)|\,dx+\int_0^t(t-x)\,|f(x)|\,dx, \end{aligned}$$

where

$$h_2(t) = [|a+ct| + |b+d(t+r)|].$$

Next, if we consider $e^{-\alpha t}|f(t)| < M_1$, $(t>0)$, we have

$$\begin{aligned}\int_0^t (t-x)|f(x)|dx &\le \int_0^t M_1(t-x)e^{\alpha x}dx \\ &= \frac{M_1}{\alpha^2}[e^{\alpha t} - 1 - \alpha t] \\ &\le \frac{M_1 e^{\alpha t}}{\alpha^2}.\end{aligned}$$

Thus, we can write

$$|u(t)| \le |h_1(t)| + \frac{M_1 e^{\alpha t}}{\alpha^2} + h_2(t)\int_0^t |u(x)|dx. \tag{3.4}$$

On the other hand,

$$|h_1(t)| \le (1+|a|)|\varphi(0)| + \int_{-r}^0 [|b| + |d|(|\tau| + r)]|\varphi(\tau)|d\tau + t[|\gamma| + |b||\varphi(-r)| + |d|\int_{-r}^0 |\varphi(\tau)|d\tau]$$

and

$$h_2(t) \le |a| + |b+dr| + [|c| + |d|]t.$$

Using the Gronwal's inequality in Theorem 2.5 in eq. (3.4) we get,

$$\begin{aligned}|u(t)| &\le |h_1(t)| + \frac{M_1 e^{\alpha t}}{\alpha^2} + h_2(t)\int_0^t \left[|h_1(\tau)| + \frac{M_1 e^{\alpha\tau}}{\alpha^2}\right] e^{\int_\tau^t h_2(\xi)d\xi} d\tau \\ &\le A_1 + B_1 t + \frac{M_1 e^{\alpha t}}{\alpha^2} + (A_2 + B_2 t)\int_0^t \left[(A_1 + B_1\tau) + \frac{M_1 e^{\alpha\tau}}{\alpha^2}\right] e^{\int_\tau^t (A_1 + B_1\xi)d\xi} d\tau \\ &\le A_1 + B_1 t + \frac{M_1 e^{\alpha t}}{\alpha^2} + (A_2 + B_2 t)\int_0^t \left[(A_1 + B_1\tau) + \frac{M_1 e^{\alpha\tau}}{\alpha^2}\right]\left[e^{A_1 t + B_1\frac{t^2}{2}} - e^{A_1\tau + B_1\frac{\tau^2}{2}}\right] d\tau \\ &\le A_1 + B_1 t + \frac{M_1 e^{\alpha t}}{\alpha^2} + (A_2 + B_2 t)e^{A_1 t + B_1\frac{t^2}{2}}\int_0^t \left[(A_1 + B_1\tau) + \frac{M_1 e^{\alpha\tau}}{\alpha^2}\right] d\tau \\ &\le A_1 + B_1 t + \frac{M_1 e^{\alpha t}}{\alpha^2} + (A_2 + B_2 t)e^{A_1 t + B_1\frac{t^2}{2}}\left[\left(A_1 t + B_1\frac{t^2}{2}\right) + \frac{M_1(e^{\alpha t} - 1)}{\alpha^3}\right],\end{aligned}$$

where A_1, B_1, A_2, B_2 are constants given as

$$\begin{aligned}A_1 &= (1+|a|)\left|\varphi(0)\right| + \int_{-r}^0 [|b| + |d|(|\tau| + r)]\left|\varphi(\tau)\right| d\tau, \\ B_1 &= \left|\gamma\right| + |b|\left|\varphi(-r)\right| + |d|\int_{-r}^0 \left|\varphi(\tau)\right| d\tau, \\ A_2 &= |a| + |b+dr|, B_2 = |c| + |d|.\end{aligned}$$

Similar results for $u'(t)$ and $u''(t)$ can be easily proved from (3.1) and (1.1), respectively. □

Theorem 3.2. *Let $\varphi(t)$, $\varphi'(t)$ are continuous on $[-r,0]$ and $F(s)$ is the Laplace transformation of $f(t)$ in* (1.1)*. Then, the exact solution of* (1.1)-(1.2)

$$u(t) = \mathscr{L}^{-1}\left\{\frac{F(s) + T(s)}{K(s)}\right\},$$

where

$$K(s) = s^2 + as + c + (bs + d)e^{-sr},$$
$$T(s) = \gamma + [s + a + be^{-sr}]\varphi(0) - b\overline{\overline{\varphi}}(s) - d\overline{\varphi}(s),$$
$$\overline{\varphi}(s) = \int_{-r}^{0} e^{-s(t+r)}\varphi(t)dt, \quad \overline{\overline{\varphi}}(s) = \int_{-r}^{0} e^{-s(t+r)}\varphi'(t)dt.$$

Proof. To solve the problem (1.1)-(1.2) by using the Laplace transform method, we recall that the Laplace transforms of the derivatives of $u(t)$ are given by (2.2)

$$\mathscr{L}\{u'(t)\} = s\mathscr{L}\{u(t)\} - \varphi(0)$$

and

$$\mathscr{L}\{u''(t)\} = s^2\mathscr{L}\{u(t)\} - s\varphi(0) - \gamma.$$

The Laplace transform for $u(t-r)$, using the definition (2.1), we give

$$\mathscr{L}\{u(t-r)\} = \int_{0}^{\infty} e^{-st}u(t-r)dt.$$

After replacing integral variable by $t = x + r$ we find that

$$\begin{aligned}\mathscr{L}\{u(t-r)\} &= \int_{-r}^{\infty} e^{-s(x+r)}u(x)dx\\ &= \int_{-r}^{0} e^{-s(x+r)}\varphi(x)dx + e^{-sr}\int_{0}^{\infty} e^{-sx}u(x)dx.\end{aligned}$$

Thus, we have

$$\mathscr{L}\{u(t-r)\} = \overline{\varphi}(s) + e^{-sr}\mathscr{L}\{u(t)\}.$$

Similarly, the Laplace transformation for $u'(t-r)$, we can write

$$\begin{aligned}\mathscr{L}\{u'(t-r)\} &= \int_{0}^{\infty} e^{-st}u'(t-r)dt\\ &= \int_{-r}^{\infty} e^{-s(x+r)}u'(x)dx\\ &= \int_{-r}^{0} e^{-s(x+r)}\varphi'(x)dx + e^{-sr}\int_{0}^{\infty} e^{-sx}u'(x)dx.\end{aligned}$$

Here, we have

$$\mathscr{L}\{u'(t-r)\} = \overline{\overline{\varphi}}(s) + e^{-sr}\mathscr{L}\{u'(t)\}.$$

Applying the Laplace transform to both sides of (1.1) gives

$$\mathscr{L}\{u''(t)\} + a\mathscr{L}\{u'(t)\} + b\mathscr{L}\{u'(t-r)\} + c\mathscr{L}\{u(t)\} + d\mathscr{L}\{u(t-r)\} = \mathscr{L}\{f(t)\}$$

and using above equalities, it can be reduced to

$$\mathscr{L}\{u(t)\} = \frac{F(s) + T(s)}{K(s)}.$$

Next, if we use inverse Laplace transform, we obtain the exact solution of (1.1)-(1.2). □

4. Illustrations

In this section, we present two particular examples that confirm the results obtained.

Example 4.1. We consider the following problem:

$$u''(t)-3u'(t)+u'(t-1)+2u(t)-u(t-1)=0, \quad t>0$$

subject to the interval condition,

$$u(t)=e^t, \ -1\le t\le 0, \ u'(0)=1.$$

If we take into consideration

$$F(s)=0, \ T(s)=s-2+e^{-s}, \ K(s)=s^2-3s+2+(s-1)e^{-s}$$

in Theorem 2.2, we easily get

$$\mathscr{L}\{u(t)\}=\frac{s-2+e^{-s}}{s^2-3s+2+(s-1)e^{-s}}=\frac{1}{s-1}$$

and

$$u(t)=\mathscr{L}^{-1}\{\frac{1}{s-1}\}=e^t.$$

Example 4.2. We consider the another problem:

$$u''(t)-3u'(t)+u'(t-1)+2u(t)-u(t-1)=1, \quad t>0$$

subject to the interval condition,

$$u(t)=e^t, \ -1\le t\le 0, \ u'(0)=1.$$

Also, if we take account of

$$F(s)=\frac{1}{s}, \ T(s)=s-2+e^{-s}, \ K(s)=s^2-3s+2+(s-1)e^{-s}$$

in Theorem 2.2, we obtain

$$\mathscr{L}\{u(t)\}=\frac{1}{s(s-1)(s-2+e^{-s})}+\frac{1}{s-1}$$

and

$$u(t)=\mathscr{L}^{-1}\left\{\frac{1}{s(s-1)(s-2+e^{-s})}+\frac{1}{s-1}\right\}.$$

Here, inverse Laplace transformation is not always easy to find. This may be considered as another subject of study.

5. Conclusion

In this study, the Laplace transformation method is applied to solve the linear second order DDE. This method is a clear and efficient technique to find the analytical solutions for the wide range of differential equations. Therefore, the results of the presented method can be extended to solve problems such as neutral delay type and Volterra delay integro differential type.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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