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Research Article

A Uniformly Convergent Numerical Study on Bakhvalov-Shishkin Mesh for Singularly Perturbed Problem

Derya Arslan

Department of Mathematics, Faculty of Art and Science, Bitlis Eren University, 13200, Bitlis, Turkey
ayredlanu@gmail.com

Abstract. In this paper, singularly perturbed multipoint boundary value problem with a right boundary layer is considered. This problem is discretized using finite difference method on Bakhvalov-Shishkin type mesh. We give uniform error estimate in a discrete maximum norm. The first-order of accuracy difference schemes for the approximate solutions of the problem are presented. The obtained numerical results demonstrate that the convergence rate of difference scheme is in accord with the theoretical analysis which means that the theoretical results are fairly sharp.

Keywords. Singular perturbation; Finite difference scheme; Bakhvalov-Shishkin mesh; Uniformly convergence; Multipoint boundary condition; Discrete maximum norm

MSC. 65L10; 65L11; 65L12; 65L15; 65L20; 65L70; 34B10

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1. Introduction

In this study, we solve the following singularly perturbed equation, which has a boundary layer at $x = 1$, with help of finite difference method based on Bakhvalov-Shishkin mesh:

$$-\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x), \quad 0 < x < 1, \tag{1}$$

equipped with the multipoint boundary conditions

$$u(0) = A, \tag{2}$$

$$u(1)-\sum_{i=1}^{m-2} c_i u(s_i)=B, \tag{3}$$

where $0<\varepsilon \ll 1$ is a very small perturbation parameter, B and c_i are given constants; $0<s_1<s_2<\ldots<s_{m-2}<1$, $i=1,2,\ldots,m-2$; $a(x)\geq\alpha>0$, $b(x)$ and $f(x)$ are assumed to be continuous functions in $[0,1]$.

Singularly perturbed equations belong to class of ordinary differential equations in which the highest derivative is multiplied by a small parameter. Also, singularly perturbed differential equations see usually in fluid mechanics and other branches of applied mathematics [2, 4, 9, 10, 17, 18, 26, 27] and the references cited therein. The first time, nonlocal boundary value problems have been defined as different cases by Bitsadze and Samarskii [10]. Nonlocal boundary value problems have also been examined seriously in the literature [1, 3, 5–8, 11–16, 18, 19] such as Amiraliyev and Cakir [3] studied to solve reaction-diffusion singularly perturbed problem with nonlocal boundary condition, Cakir and Amiraliyev [15] gave uniform finite difference method on piecewise uniform Shishkin type mesh for solving singularly perturbed three-point boundary value problem. Cakir [5] proposed hybrid scheme on Shishkin mesh for solving singularly perturbed boundary value problem with nonlocal boundary condition. The study of existence and uniqueness of these problems can be seen in [21–23].

Because of the ε-perturbation parameter, standard discretization methods for these singularly perturbed problems create instability. Therefore, we can propose suitable numerical methods such as finite difference method, finite element method, etc. We solve singularly perturbed convection-diffusion problem with multi-point condition using finite difference method in this study as well. Bakhvalov-Shishkin mesh is a modification of the Shishkin mesh described that incorporates idea by Bakhvalov. But the original Bakhvalov mesh requires the solution of a nonlinear equation to determine the transition point where the mesh switches from coarse to fine. Instead, the transition points are as in the Shishkin mesh [24]. There are many studies on the B-S (Bakhvalov-Shishkin) mesh [20, 24, 25, 28, 29].

This study is prepared as follows: Properties of the exact solution and its derivation will be determined in Section 2. In Section 3, the finite difference method will be presented. The reminder terms will be evaluated in Section 4. In Section 5, the results of numerical experiment will be presented. These will be shown by table and figures.

Henceforth, in the paper, C and C_0 will mean a positive constant independent of ε and the mesh parameter.

2. Properties of the Exact Solution

Herein we will give important properties of the solution of (1)-(3), which are needed in next sections for the examination of numerical solution.

Lemma 1. *If $a(x)$, $b(x)$ and $f(x)$ be sufficiently smooth functions on interval $[0,1]$ and*

$$w(1)-[c_1w(s_1)+c_2w(s_2)+c_3w(s_3)]\neq 0, \tag{4}$$

where $w(x)$ is the solution of the following problem:

$$-\varepsilon w''+a(x)w'(x)+b(x)w(x)=0,$$
$$w(0)=0,\ w(1)=1.$$

Then, for the solution $u(x)$ of the problem (1)-(3) *the following estimates hold:*

$$\|u(x)\|_{C[0,1]} \le C_0, \tag{5}$$

and

$$|u'(x)| \le C\left\{1+\frac{1}{\varepsilon}(e^{-\frac{\alpha(1-x)}{\varepsilon}})\right\}, \quad 0<x<1. \tag{6}$$

Proof. We take $u(1)=\lambda$ and $u(x)$ solution of (1) as $u(x)=v(x)+\lambda w(x)$, where

$$\lambda = \frac{b-v(1)+\sum_{i=1}^{m-2} c_i v(s_i)}{w(1)-\sum_{i=1}^{m-2} c_i w(s_i)},$$

and, the functions $v(x)$ and $w(x)$ is the solution of the following problems:

$$Lv = f(x),$$
$$v(0)=A,\ v'(0)=0,$$
$$Lw = 0,$$
$$w(0)=0,\ w'(0)=1.$$

After using the Maximum Principle for the above problems, we deduce the evaluations as

$$|v(x)| = |v(0)|+|v'(0)|+\alpha^{-1}\|f(x)\|_{C[0,1]} \le C, \tag{7}$$

and

$$|w(x)| = |w(0)|+|w'(0)| \le 1. \tag{8}$$

Ultimately, from (7) and (8), we have

$$|u(x)| = |v(x)|+|\lambda||w(x)| \le C+1 \le C_0.$$

This result show us the estimation (5).

Let us prove (6) as follows:

Initially, we take as $u'(x)=v(x)$ and $G(x)=f(x)-b(x)u(x)$ in equation (1), then rewrite (1) for proving (6) as

$$-\varepsilon v'(x)+a(x)v(x)=G(x),$$

and we give solution of this equation

$$v(x)=u'(x)=e^{\frac{1}{\varepsilon}\int_0^x a(\xi)d\xi}\left[u'(0)-\int_0^x G(\tau)e^{\frac{-1}{\varepsilon}\int_\tau^x a(\eta)d\eta}d\tau\right].$$

After this equation is integrated over $(0,x)$ and some arrangements, it is obtained that

$$|u'(x)| \le C+\frac{C}{\varepsilon}(e^{-\frac{\alpha(1-x)}{\varepsilon}}) \le C\left\{1+\frac{1}{\varepsilon}(e^{-\frac{\alpha(1-x)}{\varepsilon}})\right\}.$$

Finally, the proof of Lemma 1 is obtained. □

3. Mesh and Generating of the Difference Scheme

In this section, we will define the well-known Bakhvalov mesh and then construct difference scheme for the problem (1)-(3).

Bakhvalov-Shishkin Mesh. For the positive even integer discretization parameter N, we divide the interval $[0,1]$ into the two subintervals $[0,1-\sigma]$ and $[1-\sigma,1]$. In practice, we usually has $\sigma << 1$. Here σ is transition point which is called as following:

$$\sigma = \min\left\{\frac{1}{2}, \alpha^{-1}\varepsilon \ln N\right\}. \tag{9}$$

We define a set of the mesh points $\bar{\omega}_N = \{x_i\}_{i=0}^{N}$ as

$$x_i = \begin{cases} ih,\ h = \frac{2(1-\sigma)}{N},\ x_i \in [0,1-\sigma],\ i = 1,\dots,\frac{N}{2}; \\ 1+\alpha^{-1}\varepsilon \ln\left[1-2(1-N^{-1})\left(1-\frac{i}{N}\right)\right],\ \sigma < \frac{1}{2}, \\ x_i \in [1-\sigma,1],\ i = \frac{N}{2}+1,\dots,N. \end{cases}$$

Difference Scheme. In what follows, we denote by ω_N nonuniform mesh and define the following finite difference for any mesh function $g_i = g(x_i)$ given on $\bar{\omega}_N$:

$$\omega_N = \{0 < x_1 < x_2 < \ldots < x_{N-1} < 1\},\quad \bar{\omega}_N = \omega_N \cup \{x_0 = 0, x_N = 1\},$$

and

$$g_{\bar{x},i} = \frac{g_i - g_{i-1}}{h_i},\quad g_{x,i} = \frac{g_{i+1} - g_i}{h_{i+1}},\quad g_{\overset{0}{x},i} = \frac{g_{x,i} + g_{\bar{x},i}}{2},$$

$$g_{\hat{x},i} = \frac{g_{i+1} - g_i}{\hbar_i},\quad g_{\bar{x}\hat{x},i} = \frac{g_{x,i} - g_{\bar{x},i}}{\hbar_i},\quad \hbar_i = \frac{h_i + h_{i+1}}{2},\quad h_i = x_i - x_{i-1},$$

$$\|g\|_\infty \equiv \|g\|_{\infty,\bar{\omega}_N} := \max_{0\le i\le N} |g_i|.$$

To obtain difference approximation for (1), we integrate (1) over (x_{i-1}, x_{i+1}):

$$\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} Lu(x)\varphi_i(x)dx = \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} f(x)\varphi_i(x)dx,\quad i = 1,\dots,N-1. \tag{10}$$

The relation (10) can be rewritten as

$$\varepsilon\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} u'(x)\varphi_i'(x)dx + a_i\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} u'(x)\varphi_i(x)dx + b_iu_i = f_i + R_{a,i} + R_{b,i}, \tag{11}$$

which yields relation

$$-\varepsilon\theta_i u_{\bar{x}\hat{x},i} + \eta_i u_{\hat{x},i} + b_iu_i = f_i + R_{a,i} + R_{b,i} = R_i,\quad i = 1,\dots,N-1, \tag{12}$$

where $R_i = f_i + R_{a,i} + R_{b,i}$, and the functions $\{\varphi_i(x)\}_{i=1}^{N-1}$ have the form

$$\varphi_i(x) = \begin{cases} \varphi_i^{(1)}(x) = \dfrac{e^{\frac{a_i(x-x_{i-1})}{\varepsilon}}-1}{e^{\frac{a_ih_i}{\varepsilon}}-1}, & x_{i-1} < x < x_i, \\ \varphi_i^{(2)}(x) = \dfrac{1-e^{\frac{a_i(x-x_{i+1})}{\varepsilon}}}{1-e^{-\frac{a_ih_{i+1}}{\varepsilon}}}, & x_i < x < x_{i+1}, \\ 0, & x \notin (x_{i-1}, x_{i+1}), \end{cases}$$

where

$$\theta_i = \frac{\frac{a_ih_i}{\varepsilon}}{1-e^{-\frac{a_ih_i}{\varepsilon}}}, \tag{13}$$

$$\eta_i = \frac{-a_ih_i}{h_{i+1}[1-e^{-\frac{a_ih_i}{\varepsilon}}]} + \frac{a_i}{1-e^{\frac{a_ih_{i+1}}{\varepsilon}}}. \tag{14}$$

With the local truncation error

$$R_i = \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} [a(x_i)-a(x)]u'(x)\varphi_i(x)dx + \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} [b(x_i)-b(x)]u(x)\varphi_i(x)dx + \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} [f(x)-f(x_i)]\varphi_i(x)dx. \tag{15}$$

Thus, by neglecting R_i in the equation (12), we suggest the following difference scheme for approximating (1)-(3):

$$-\varepsilon\theta_i y_{\bar{x}\hat{x},i} + \eta_i y_{\hat{x},i} + b_i y_i = f_i, \quad i = 1,\ldots,N-1, \tag{16}$$

$$y_0 = A, \tag{17}$$

$$y_N = \sum_{i=1}^{m-2} c_i y_{N_i}\left(x_{N_i}\right) + B, \tag{18}$$

where x_{N_i} are the mesh points nearest to s_i, and also θ_i and η_i are given by (13) and (14), respectively.

4. Convergence Analysis

Let $z_i = y_i - u_i$, $i = 0,1,\ldots,N$. Then, the error in the numerical solution satisfies where the truncation error R_i is given by (15)

$$-\varepsilon\theta_i z_{\bar{x}\hat{x},i} + \eta_i z_{\hat{x},i} + b_i z_i = -R_i, \quad i = 1,\ldots,N-1, \tag{19}$$

$$z_0 = 0, \tag{20}$$

$$z_N = \sum_{i=1}^{m-2} c_i z_{N_i}. \tag{21}$$

Lemma 2. *Let z_i be the solution of problem* (19)-(21). *Then, the estimate*

$$\|z\|_{\infty,\bar{\omega}_N} \le C\,\|R\|_{\infty,\omega_N}, \tag{22}$$

holds.

Proof. According to the maximum principle, we have the following inequalities:

$$w(x) = \pm z_i + \alpha^{-1}\|R\|_{\infty,\omega_N}, \tag{23}$$

$$w(0) = \pm z_0 + \alpha^{-1}\|R\|_{\infty,\omega_N} \ge 0, \tag{24}$$

and

$$w(1) = \pm z_N + \alpha^{-1}\|R\|_{\infty,\omega_N} \ge 0. \tag{25}$$

From (23)-(25), we obtain that

$$\|z_i\| \le \alpha^{-1}\|R\|_{\infty,\omega_N} \le C\|R\|_{\infty,\omega_N},$$

which proves Lemma 2. □

Lemma 3. *If $a(x), b(x), f(x) \in C^1[0,1]$, then for the truncation error R_i we have*

$$|R_i| \le CN^{-1}. \tag{26}$$

Proof. We can rewrite for the truncation error R_i as

$$|R_i| \le \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} |a(x_i) - a(x)|u'(x)\varphi_i(x)dx + \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} |b(x_i) - b(x)|u(x)\varphi_i(x)dx$$
$$+\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} |f(x) - f(x_i)|\varphi_i(x)dx. \tag{27}$$

Using the mean value theorem for $|a(x_i)-a(x)|$, $|b(x_i)-b(x)|$ and $|f(x)-f(x_i)|$ in (27), we get

$$|a(x)-a(x_i)| = |a'(\xi)|\,|x-x_i| \le Ch_i,$$
$$|b(x_i)-b(x)| = |b'(\xi)|\,|x-x_i| \le Ch_i,$$
$$|f(x)-f(x_i)| = |f'(\xi)|\,|x-x_i| \le Ch_i, \quad \xi\in[x_i,x],\ i=0,\dots,N,$$

and also we evaluate (6) as follows:

$$|u'(x)| \le C\left\{h_i + e^{-\frac{\alpha(1-x_{i+1})}{\varepsilon}} - e^{-\frac{\alpha(1-x_{i-1})}{\varepsilon}}\right\} \le Ch_i, \quad i=1,\dots,N-1,$$

where

$$e^{-\frac{\alpha(1-x_{i+1})}{\varepsilon}} - e^{-\frac{\alpha(1-x_{i-1})}{\varepsilon}} \le e^{-\frac{\alpha(1-x_{i+1})}{\varepsilon}}\left(1-e^{-\frac{\alpha(x_{i+1}-x_{i-1})}{\varepsilon}}\right) \le Ch_i.$$

From here with (6) and (27), we have

$$|R_i| \le Ch_i, \quad i=0,\dots,N. \tag{28}$$

Now, we can begin to evaluate for (27) on the intervals $[0,1-\sigma]$ and $[1-\sigma,1]$, respectively. In the first case $x_i\in[0,1-\sigma]$:

$$x_i = \sigma + \left(i-\frac{N}{2}\right)h, \quad i=0,\dots,\frac{N}{2},$$

where

$$\alpha^{-1}\varepsilon\ln N < \frac{1}{2}, \quad h = \frac{2(1-\alpha^{-1}\varepsilon\ln N)}{N} \le CN^{-1}. \tag{29}$$

It then follows from (27) and (29), we have

$$h_i = h, \ \ |R_i| \le Ch \le CN^{-1}. \tag{30}$$

In the second case $x_i\in[1-\sigma,1]$:
For $\sigma<\frac{1}{2}$,

$$x_{i-1} = 1+\alpha^{-1}\varepsilon\ln\left[1-2(1-N^{-1})\left(1-\frac{i-1}{N}\right)\right], \tag{31}$$

$$h_i = \alpha^{-1}\varepsilon\ln\left[1-2(1-N^{-1})\left(1-\frac{i}{N}\right)\right] - \alpha^{-1}\varepsilon\ln\left[1-2(1-N^{-1})\left(1-\frac{i-1}{N}\right)\right]. \tag{32}$$

Applying the mean value theorem in (32), we obtain that

$$h_i = \alpha^{-1}\varepsilon\frac{2(1-N^{-1})N^{-1}}{1-2i_1(1-N^{-1})N^{-1}} \le CN^{-1}. \tag{33}$$

Thus, from (27) and (33), we can write

$$|R_i| \le CN^{-1}, \quad i=\frac{N}{2}+1,\dots,N.$$

According to all these situations, we have

$$|R_i| \le CN^{-1}, \quad i=0,\dots,N. \qquad \square$$

Now, we can formulate the main convergence result:

Theorem 1. *Let $u(x)$ be the solution of the problem* (1)-(3) *and y_i be the solution of the difference scheme* (16)-(18). *Then, the following uniform error estimate satisfies*

$$\|y-u\|_{\infty,\bar{\omega}_N} \le CN^{-1}.$$

5. Numerical Results

In this section we will solve a convection-diffusion problem with the Bakhvalov-Shishkin mesh. Firstly, we construct an algorithm and then using a computer program, we obtain numarical results, table and figures.

Example 2. Consider a convection-diffusion problem:

$$-\varepsilon u''(x)+u'(x)=1, \quad 0<x<1,$$

$$u(0)=0, \quad u(1)=u(0.25)+2u\left(\frac{1}{3}\right)+3u(0.5)+d.$$

The exact solution is

$$u(x)=\frac{\exp\left(-\frac{x}{\sqrt{\varepsilon}}\right)+\exp\left(\frac{x-1}{\sqrt{\varepsilon}}\right)}{1+\exp\left(-\frac{1}{\sqrt{\varepsilon}}\right)}-\cos^2(\pi x).$$

The corresponding ε-uniform convergence rates are computed using the formula

$$P^N=\frac{\ln\left(e^N/e^{2N}\right)}{\ln 2}.$$

The error estimates are denoted by

$$e^N=\max_{\varepsilon} e_{\varepsilon}^N, \quad e_{\varepsilon}^N=\|y-u\|_{\infty,\bar{\omega}_N}.$$

Table 1. The computed maximum pointwise errors e^N and rates of convergence p^N of Example.

ε	$N=24$	$N=48$	$N=96$	$N=192$	$N=384$	$N=768$
2^{-10}	0.04166666	0.02083333	0.01041667	0.00520785	0.00255726	0.00098445
	0.99	0.99	1.00	1.00	1.31	
2^{-12}	0.04166665	0.02083334	0.01041667	0.00520832	0.00260416	0.00130196
	0.99	1.00	1.00	0.99	1.00	
2^{-14}	0.04166667	0.02083338	0.01041666	0.00520832	0.00260417	0.00130207
	0.99	1.00	1.00	0.99	1.00	
2^{-16}	0.04166626	0.02083325	0.01041647	0.00520834	0.00260414	0.00130207
	0.99	1.00	0.99	1.00	0.99	
2^{-18}	0.04166681	0.02083366	0.01041777	0.00520797	0.00260409	0.00130199
	0.99	1.00	0.99	0.99	1.00	
2^{-20}	0.04166353	0.02083475	0.01041231	0.00520910	0.00260409	0.00130169
	0.99	1.00	0.99	1.00	1.00	
e^N	0.04166681	0.02083475	0.01041777	0.00520910	0.00260409	0.00130207
p^N	0.99	0.99	0.99	0.99	0.99	

The resulting errors and the corresponding numbers for $\varepsilon=2^{-2i}$, $i=5,6,7,8,9,10$ are listed in Table 1. Table 1 verifies first-order the ε-uniform convergence of the numerical solution on both subintervals and computed rates are essentially in agreement with our theoretical analysis.

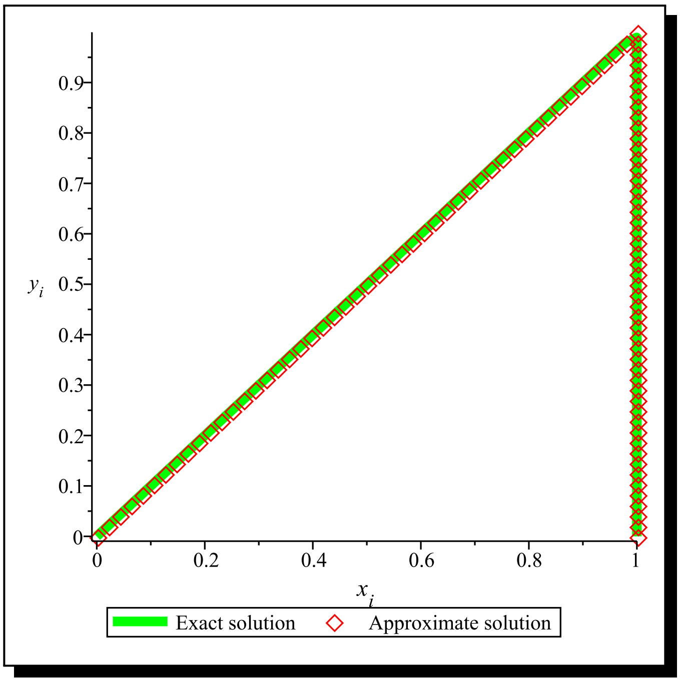

Figure 1. Comparison of approximate solution and exact solution of Example for $N = 96$, $\varepsilon = 2^{-14}$.

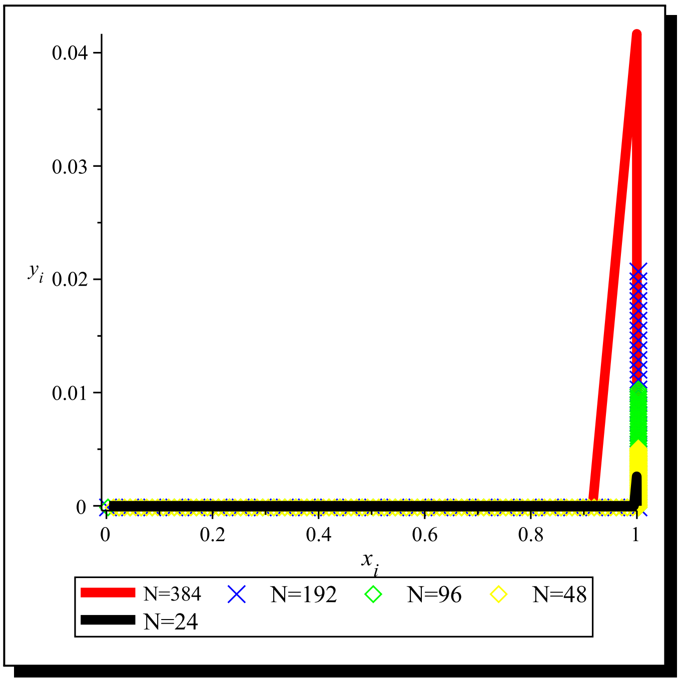

Figure 2. Error distribution of Example for $\varepsilon = 2^{-16}$.

The exact solution and approximate solution curves are almost identical as shown in Figure 1. In Figure 2, the errors in boundary layer region are maximum because of the irregularity caused by the sudden and rapid change of the solution in the layer region around $x = 1$ for different ε and N values.

6. Conclusion

In this work, the finite difference scheme is proposed to compute the solution of singularly perturbed problems with multipoint boundary condition. It is proved that the error estimate of numerical solution is first-order on Bakhvalov-Shishkin mesh. The errors and rates of convergence are tabulated in Table 1 for the considered example in support of the theoretical results. The figures of the exact and the numerical solution of the problem for different values of ε-perturbation parameter were plotted in Figure 1. In Figure 2, error distributions of Example for $\varepsilon = 2^{-16}$, $N = 24, 48, 96, 192, 384, 768$ are plotted. Thus, we say that this study can solve the singularly perturbed problems with more complicated multipoint boundary condition.

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Competing Interests

The author declares that she has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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