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Research Article

Numerical Approach for Differential-Difference Equations with Layer Behaviour

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Abstract. A numerical scheme is proposed using a non polynomial spline to solve the differential-difference equations having layer behaviour, with delay as well advanced terms. The retarded terms are handled by using Taylor's series, subsequently the given problem is substituted by an equivalent second order singular perturbation problem. A finite difference scheme using non polynomial spline is derived and it is applied to the singular perturbation problem using non standard differences of the first derivatives. Tridiagonal algorithm is used to solve the resulting system. The method is exemplified on numerical examples with various values of perturbation, delay and advance parameters. Maximum absolute errors are computed and tabulated to support the method. Numerical solutions are pictured in graphs and the effects of small shifts on the boundary layer region has been investigated. Also, the convergence of the proposed method has also been established.

Keywords. Differential-difference equations; Boundary Layer; Non polynomial spline; Maximum absolute error

MSC. 65L10; 65L11; 65L12

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1. Introduction

The problems of differential-difference comes about in the modelling of many practical phenomena such as, thermo-elasticity [2], hybrid optical system [3], in population dynamics [10], in models for physiological processes [14], red blood cell system [13], predetor-prey models [15] and in the potential in nerve cells by random synaptic inputs in dendrites [18].

For further study of mathematical aspects of the above class of models and singular perturbation problems, one can be found in the collection of books, to name a few, Bellman and Cooke [1], Doolan et al. [5], Driver [6], El'sgol'ts and Norkin [7], Kokotovic [9], Miller et al. [16] and Smith [17].

Lange and Miura [11, 12] presented an analysis of differential-difference equations with small shifts, layers having turning points and rapid oscillations. In [4], a fourth order finite difference method with a fitting factor is proposed for the solution of the singularly perturbed differential-difference equations with mixed shifts. The authors in [8], proposed a fitted piecewise-uniform mesh method with analysis for differential difference equation having mixed small shifts having boundary layer. With this motivation, in the next section, we describe the problem and derivation of the of the numerical scheme using non polynomial spline.

2. Numerical Approach

Consider the singularly perturbed differential-difference equation with small delay as well as advance terms of the form:

$$\varepsilon w''(t)+a(t)w'(t)+b(t)w(t-\delta)+c(t)w(t)+d(t)w(t+\eta)=f(t), \quad 0<t<1 \tag{1}$$

subject to the interval and boundary conditions

$$w(t)=\phi(t), \quad \text{on} \quad -\delta\le t\le 0 \tag{2}$$

$$w(t)=\gamma(t), \quad \text{on} \quad 1\le t\le 1+\eta \tag{3}$$

where $a(t)$, $b(t)$, $c(t)$, $d(t)$, $\phi(t)$ and $\gamma(t)$ are bounded and continuously differentiable functions on $(0,1)$, ε is the small perturbation parameter $(0<\varepsilon\ll 1)$, δ and η are the delay and the advance parameters respectively $(0<\delta=o(\varepsilon);\ 0<\eta=o(\varepsilon))$.

Applying Taylor series on retarded terms in the neighbourhood of the point t, we have

$$w(t-\delta)\approx w(t)-\delta w'(t), \tag{4}$$

$$w\left(t+\eta\right)\approx w(t)+\eta w'(t), \tag{5}$$

Using eqs. (4) and (5) in eq. (1), we get an equivalent second order singular perturbation problem of the form:

$$\varepsilon w''(t)+p(t)w'(t)+q(t)w(t)=f(t) \tag{6}$$

with boundary conditions

$$w(0)=\phi(0), \tag{7}$$

$$w(1)=\gamma(1), \tag{8}$$

where $p(t)=a(t)+d(t)\eta-b(t)\delta$ and $q(t)=b(t)+c(t)+d(t)$.

Since $0<\delta\ll 1$ and $0<\eta\ll 1$, the transition from eq. (1) to eq. (6) is admitted. This replacement is significant from the computational point of view (El'sgolt's and Norkin [3]).

2.1 Non polynomial spline

To construct the difference equation of the problem eqs. (6)-(8), the domain $[0,1]$ is divided into N non overlapping intervals $0=t_0<t_1<t_2<\ldots<t_N=1$, each of length h. Then, we have

$t_i = t_0 + ih$ for $i = 0,1,\ldots,N$. For simplicity, denote$A(t_i) = A_i$, $B(t_i) = B_i$, $f(t_i) = f_i$, $w(t_0) = w_0$, $w(t_i) = w_i$, $w(t_i + h) = w_{i+1}$, $w(t_i - h) = w_{i-1}$, $w'(t_i) = w'_i$, $w''(t_i) = w''_i$, etc.

For each ith segment, the cubic non-polynomial spline function $P_i(t)$ has the form

$$P_i(t) = \widetilde{a}_i + \widetilde{b}_i(t - t_i) + \widetilde{c}_i \sin\tau(t - t_i) + \widetilde{d}_i \cos\tau(t - t_i), \quad i = 0,1,\ldots,N-1, \tag{9}$$

where $\widetilde{a}_i$, $\widetilde{b}_i$, $\widetilde{c}_i$ and $\widetilde{d}_i$ are constants and τ is a free parameter.

Let $w(t)$ be the exact solution and w_i be an approximation to $w(t_i)$ obtained by the non polynomial cubic spline $P_i(t)$ passing through the points (t_i, w_i) and (t_{i+1}, w_{i+1}).

The spline $P_i(t)$ satisfies interpolatory conditions at t_i and t_{i+1}, also the continuity of first derivative at the common nodes (t_i, w_i). The non-polynomial function $P(t)$ of class $C^2[A,B]$ interpolates $w(t)$ at the grid points t_i, for $i = 0,1,\ldots,N$, depends on a parameter τ, and reduces to an ordinary cubic spline $P(t)$ in $[A,B]$ as $\tau \to 0$.

To derive the expressions for the unknown coefficients of eq. (9) in terms of w_i, w_{i+1}, M_i and M_{i+1}, define

$$P_i(t_i) = w_i, \quad P_i(t_{i+1}) = w_{i+1},$$
$$P''_i(t_i) = M_i, \; P''_i(t_{i+1}) = M_{i+1}.$$

Then by algebraic calculations, we get the following expressions:

$$\widetilde{a}_i = w_i + \frac{M_i}{\tau^2}, \quad \widetilde{b}_i = \frac{w_{i+1} - w_i}{h} + \frac{M_{i+1} - M_i}{\tau\theta}, \quad \widetilde{c}_i = \frac{M_i \cos\theta - M_{i+1}}{\tau^2 \sin\theta}, \quad \widetilde{d}_i = -\frac{M_i}{\tau^2},$$

where $\theta = \tau h$, for $i = 0,1,2,\ldots,N-1$.

Using the continuity of the first derivative at (t_i, w_i), that is $P'_{i-1}(t_i) = P'_i(t_i)$,we arrived at the following relation:

$$\alpha M_{i+1} + 2\beta M_i + \alpha M_{i-1} = \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} \quad \text{for } i = 1,2,\ldots,N-1, \tag{10}$$

where

$$\alpha = \frac{-1}{\theta^2} + \frac{1}{\theta \sin\theta}, \; \beta = \frac{1}{\theta^2} - \frac{\cos\theta}{\theta \sin\theta}, \; M_j = w''(t_j), \quad j = i, \; i \pm 1 \text{ and } \theta = \tau h.$$

For our convenience, at the grid points x_i, rearranging eq. (1) we obtain:

$$\varepsilon w''_i = \tilde{A}(t_i) w'_i + \tilde{B}(t_i) w_i + f_i,$$

where $\tilde{A}(t) = -p(t)$, $\tilde{B}(t) = -q(t)$.

By using Spline's second derivatives, we have

$$\varepsilon M_j = \tilde{A}(t_j) w'_j(t) + \tilde{B}(t_j) w(t_j) + f(t_j) \quad \text{for } j = i-1, \; i, \; i+1.$$

Substituting the above equations in eq. (10) and using the following approximations for the first order derivative of y at the grid points $t_1, t_2, \ldots, t_{N-1}$

$$w'_{i+1} \cong \frac{w_{i-1} - 4w_i + 3w_{i+1}}{2h}, \tag{11}$$

$$w'_{i-1} \cong \frac{-3w_{i-1} + 4w_i - w_{i+1}}{2h}, \tag{12}$$

$$w'_i \cong \left(\frac{1 + 2\omega h^2 \tilde{B}_{i+1} + \omega h [3\tilde{A}_{i+1} + \tilde{A}_{i-1}]}{2h} \right) w_{i+1} - 2\omega \left[\tilde{A}_{i+1} + \tilde{A}_{i-1}\right] w_i$$

$$-\left(\frac{1+2\omega h^2\tilde{B}_{i-1}-\omega h[\tilde{A}_{i+1}+3\tilde{A}_{i-1}]}{2h}\right)w_{i-1}+\omega h[f_{i+1}-f_{i-1}]. \tag{13}$$

W get the following three term relation given as:

$$E_i w_{i-1}+F_i w_i+G_i w_{i+1}=H_i \quad \text{for } i=1,2,\dots,N-1, \tag{14}$$

where

$$E_i=-\varepsilon-\frac{3}{2}\alpha\tilde{A}_{i-1}h+\beta\tilde{A}_i h^2\omega\left[\tilde{A}_{i+1}+3\tilde{A}_{i-1}\right]-2\omega\tilde{A}_i\beta h^3\tilde{B}_{i-1}+\frac{\alpha}{2}\tilde{A}_{i+1}h+\alpha\tilde{B}_{i-1}h^2-h\beta\tilde{A}_i,$$
$$F_i=2\varepsilon+2\alpha\tilde{A}_{i-1}h-4\beta\tilde{A}_i h^2\omega\left[\tilde{A}_{i+1}+\tilde{A}_{i-1}\right]-2\alpha\tilde{A}_{i+1}h+2\beta\tilde{B}_i h^2,$$
$$G_i=-\varepsilon-\frac{\alpha}{2}\tilde{A}_{i-1}h+\beta\tilde{A}_i h^2\omega\left[3\tilde{A}_{i+1}+\tilde{A}_{i-1}\right]+2\omega h^3\beta\tilde{A}_i\tilde{B}_{i+1}+\frac{3}{2}\alpha\tilde{A}_{i+1}h+\alpha\tilde{B}_{i+1}h^2+h\beta\tilde{A}_i,$$
$$H_i=-h^2\left[\left(\alpha-2\omega\beta\tilde{A}_i h\right)\tilde{f}_{i-1}+2\beta f_i+\left(\alpha+2\omega\beta\tilde{A}_i h\right)f_{i+1}\right].$$

The resulting tri-diagonal system eq. (14) is solved by Thomas algorithm, to get the approximations $y_1,y_2,\dots,y_{N-1}$ of the solution $y(t)$ at $t_1,t_2,\dots,t_{N-1}$.

3. Convergence Analysis

Now we consider the convergence analysis of the non-polynomial spline method, described in this section. Incorporating the boundary conditions we obtain the system of equations in the matrix form as

$$(D+J)Y+\tilde{R}+T(h)=O, \tag{15}$$

where

$$D=[-\varepsilon,2\varepsilon,-\varepsilon]=\begin{bmatrix} 2\varepsilon & -\varepsilon & 0 & 0 & \dots & 0\\ -\varepsilon & 2\varepsilon & -\varepsilon & 0 & \dots & 0\\ 0 & -\varepsilon & 2\varepsilon & -\varepsilon & \dots & 0\\ \dots & \dots & \dots & \dots & \dots & \\ \dots & \dots & \dots & \dots & \dots & \\ 0 & \dots & \dots & 0 & -\varepsilon & 2\varepsilon \end{bmatrix},$$

$$J=[z_i,v_i,u_i]=\begin{bmatrix} v_1 & u_1 & 0 & 0 & \dots & 0\\ z_2 & v_2 & u_2 & 0 & \dots & 0\\ 0 & z_3 & v_3 & u_3 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & \dots & \dots & z_{N-1} & v_{N-1} & \end{bmatrix},$$

$$z_i=-\frac{3}{2}\alpha\tilde{A}_{i-1}h+\beta\tilde{A}_i h^2\omega\left[\tilde{A}_{i+1}+3\tilde{A}_{i-1}\right]-2\omega\tilde{A}_i\beta h^3\tilde{B}_{i-1}+\frac{\alpha}{2}\tilde{A}_{i+1}h\alpha\tilde{B}_{i-1}h^2-h\beta\tilde{A}_i,$$
$$v_i=2\alpha\tilde{A}_{i-1}h-4\beta\tilde{A}_i h^2\omega\left[\tilde{A}_{i+1}+\tilde{A}_{i-1}\right]-2\alpha\tilde{A}_{i+1}h+2\beta\tilde{B}_i h^2,$$
$$u_i=-\frac{\alpha}{2}\tilde{A}_{i-1}h+\beta\tilde{A}_i h^2\omega\left[3\tilde{A}_{i+1}+\tilde{A}_{i-1}\right]+2\omega h^3\beta\tilde{A}_i\tilde{B}_{i+1}+\frac{3}{2}\alpha\tilde{A}_{i+1}h+\alpha\tilde{B}_{i+1}h^2+h\beta\tilde{A}_i,$$
$$\text{for } i=1,2,\dots,N-1,$$

$$\tilde{R}=\left[\tilde{B}_1+(-\varepsilon+z_1)\gamma_0,\tilde{B}_2,\tilde{B}_3,\dots,\tilde{B}_{N-1}+(-\varepsilon+w_{N-1})\gamma_1\right],$$
$$\tilde{B}_i=h^2\left[\left(\alpha-2\omega\beta\tilde{A}_i h\right)f_{i-1}+2\beta f_i+\left(\alpha+2\omega\beta\tilde{A}_i h\right)f_{i+1}\right], \quad i=1,2,\dots,N-1$$

and $W = [W_1, W_2, \dots, W_{N-1}]^T$, $T(h) = [T_1, T_2, \dots, T_{N-1}]^T$, $O = [0, 0, \dots, 0]^T$ are the associated vectors with eq. (15).

The local truncation error associated with the scheme developed in eq. (14) is

$$T(h) = \left[-1 + 2(\alpha+\beta)\right]\varepsilon h^2 w''(t_i) + \left\{\left[\left(4\omega\varepsilon + \frac{1}{3}\right)\beta - \frac{2\alpha}{3}\right]\tilde{A}(x_i)w'''(t_i) + (-1+12\alpha)\frac{\varepsilon}{12}w^{(4)}(t_i)\right\}h^4 + O(h^6)$$

i.e.,

$$T(h) = O(h^6), \quad \text{for } \alpha = \frac{1}{12}, \beta = \frac{5}{12}, \omega = -\frac{1}{20\varepsilon}.$$

Let $w = [w_1, w_2, \dots, w_{N-1}]^T \cong W$ which satisfies the equation

$$(D+J)w + \widetilde{R} = 0. \tag{16}$$

Let $e_i = w_i - W_i$, $i = 1, 2, \dots N-1$ be the discretization error so that

$$E = [e_1, e_2, \dots, e_{N-1}]^T = w - W.$$

Using eq. (15) and eq. (16), we have the error equation as:

$$(D+J)E = T(h). \tag{17}$$

Let $|\tilde{A}(x)| \le C_1$ and $|\tilde{B}(x)| \le C_2$ where C_1, C_2 are positive constants. If $J_{i,j}$ be the (i,j)th element of J, then

$$|J_{i,i+1}| = |u_i| \le (h(\alpha+\beta)C_1 + h^2\alpha C_2 + 4\beta\omega h^2 C_1^2 + 2h^3\beta\omega C_1 C_2), \quad i = 1, 2, \dots, N-2,$$

$$|J_{i,i-1}| = |z_i| \le (h(\alpha+\beta)C_1 + h^2\alpha C_2 + 4\beta\omega h^2 C_1^2 + 2h^3\beta\omega C_1 C_2), \quad i = 2, 3, \dots, N-1.$$

Thus for h,

$$|J_{i,i+1}| < \varepsilon, \quad i = 1, 2, \dots, N-2 \tag{18}$$

and

$$|J_{i,i-1}| < \varepsilon, \quad i = 2, 3, \dots, N-1. \tag{19}$$

Hence $(D+J)$ is irreducible. Let P_i be the sum of the elements of the ith row of the matrix $(D+J)$, then we have

$$\tilde{P}_i = \varepsilon + \frac{\alpha h}{2}\left(3\tilde{A}_{i-1} - \tilde{A}_{i+1}\right) - h\beta\tilde{P}_i + h^2\left(\alpha\tilde{B}_{i+1} + 2\beta\tilde{B}_i\right) - h^2\beta\omega\tilde{A}_i\left(\tilde{A}_{i+1} + 3\tilde{A}_{i-1}\right) + 2h^3\beta\omega\tilde{A}_i\tilde{B}_{i+1}, \quad \text{for } i = 1,$$

$$\tilde{P}_i = h^2\left(\alpha\tilde{B}_{i-1} + 2\beta\tilde{B}_i + \alpha\tilde{B}_{i+1}\right) + 2h^3\beta\tilde{A}_i\omega\left(\tilde{B}_{i+1} - \tilde{B}_{i-1}\right), \quad \text{for } i = 2, 3, \dots, N-2,$$

$$\tilde{P}_i = \varepsilon + \frac{\alpha h}{2}\left(\tilde{A}_{i-1} - 3\tilde{A}_{i+1}\right) - h\beta\tilde{A}_i + h^2\left(\alpha\tilde{B}_{i-1} + 2\beta\tilde{B}_i\right) - h^2\beta\omega\tilde{A}_i\left(\tilde{A}_{i+1} + \tilde{A}_{i-1}\right) - 2h^3\beta\omega\tilde{A}_i\tilde{B}_{i-1}, \quad \text{for } i = N-1.$$

Let $C_{1^*} = \min_{1\le i\le N}|\tilde{A}(t)|$ and $C_1^* = \max_{1\le i\le N}|\tilde{A}(t)|$, $C_{2^*} = \min_{1\le i\le N}|\tilde{B}(t)|$ and $C_2^* = \max_{1\le i\le N}|\tilde{B}(t)|$.
Since $0 < \varepsilon \ll 1$ and $\varepsilon \propto O(h)$, it is verify that for h, $(D+J)$ is monotone.
Hence $(D+J)^{-1}$ exists and $(D+J)^{-1} \ge 0$.
Thus from eq. (17), we have

$$\|E\| \le \|(D+J)^{-1}\| \cdot \|T\|. \tag{20}$$

Let $(D+J)^{-1}_{i,k}$ be the (i,k)th element of $(D+J)^{-1}$ and we define

$$\|(D+J)^{-1}\| = \max_{1\le i\le N-1} \sum_{k=1}^{N-1} (D+J)^{-1}_{i,k} \quad \text{and} \quad \|T(h)\| = \max_{1\le i\le N-1} |T(h)|. \tag{21}$$

Since $(D+J)^{-1}_{i,k} \ge 0$ and $\sum\limits_{k=1}^{N-1} (D+J)^{-1}_{i,k} \cdot \tilde{P}_k = 1$ for $i = 1,2,3,\dots,N-1$.

Hence

$$(D+J)^{-1}_{i,k} \le \frac{1}{P_i} < \frac{1}{h^2\left[(\alpha+2\beta)C_{2^*} - 4\beta\omega C^2_{1^*}\right]}, \quad i = 1, \tag{22}$$

$$(D+J)^{-1}_{i,k} \le \frac{1}{P_i} < \frac{1}{h^2\left[(\alpha+2\beta)C_{2^*} - 4\beta\omega C^2_{1^*}\right]}, \quad i = N-1. \tag{23}$$

Further

$$\sum_{k=1}^{N-1} (D+J)^{-1}_{i,k} \le \frac{1}{\min_{2\le i\le N-2} P_i} < \frac{1}{h^2\left(2(\alpha+\beta)C_{2^*}\right)}. \tag{24}$$

Using the eqs. (22)-(24), from eq. (20), we get

$$\|E\| \le O(h^4).$$

Hence, the method is fourth order convergent for $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$, $\omega = -\frac{1}{20\varepsilon}$ on uniform mesh.

4. Numerical Experiments

To demonstrate the proposed method computationally, we consider four boundary value problems. The maximum absolute errors in the solution are estimated by using $E_{N,\varepsilon} = \max\limits_{0\le i\le N} |w(t_i) - w_i|$, where $w(t_i)$ is the exact solution and w_i is the computed solution.

Example 1. Consider the differential-difference equation with left end boundary layer $\varepsilon w''(t) + w'(t) + 2w(t-\delta) - 3w(t) = 0$ with $w(0) = 1$, $-\delta \le t \le 0$ and $w(1) = 1$, $1 \le t \le 1+\eta$.

Table 1. Maximum errors in the solution of Example 1 for $\varepsilon = 0.1$ with different values of δ

	$N \to 8$	32	128	512
$\delta \downarrow$	Present method			
0.00	1.0798e-03	4.0303e-06	1.5642e-08	6.1089e-11
0.05	7.9692e-04	2.9371e-06	1.1431e-08	4.4649e-11
0.09	6.0682e-04	2.2124e-06	8.6762e-09	3.3887e-11
	Results in Kadalbajoo and Sharma [8]			
0.00	0.09907804	0.03700736	0.00954678	0.00214501
0.05	0.09659609	0.03640566	0.00924661	0.00202998
0.09	0.09277401	0.03556652	0.00895172	0.00192488

Example 2. Consider the equation with left end boundary layer $\varepsilon w''(t) + w'(t) - 3w(t) + 2w(t+\eta) = 0$ with $w(0) = 1$, $-\delta \le t \le 0$ and $w(1) = 1$, $1 \le t \le 1+\eta$.

Table 2. Maximum errors in the solution of Example 2 for $\varepsilon = 0.1$ with different values of η

	$N \to 8$	32	128	512
$\eta \downarrow$	Present Method			
0.00	1.0798e-03	4.0303e-06	1.5642e-08	6.1089e-11
0.05	1.4141e-03	5.3400e-06	2.0819e-08	8.1296e-11
0.09	1.7187e-03	6.5463e-06	2.5726e-08	1.0044e-10
	Results in Kadalbajoo and Sharma [8]			
0.00	0.09907804	0.03700736	0.00954678	0.00214501
0.05	0.09977501	0.03727087	0.00979659	0.00224472
0.09	0.10031348	0.03723863	0.00996284	0.00458698

Example 3. Consider the problem equation with left end boundary layer
$\varepsilon w''(t) + w'(t) - 2w(t-\delta) - 5w(t) + w(t+\eta) = 0$ with $w(t) = 1$, $-\delta \le t \le 0$ and $w(t) = 1$, $1 \le t \le 1+\eta$.

Table 3. Maximum errors in Example 3 for $\varepsilon = 0.1$ with different values of η and δ

	$N = 8$	$N = 32$	$N = 128$	$N = 512$
$\delta \downarrow$	$\eta = 0.5\varepsilon$	Present method		
0.00	3.6277e-03	1.4948e-05	5.7829e-08	2.2590e-10
0.05	4.6884e-03	1.9939e-05	7.6887e-08	3.0022e-10
0.09	5.6439e-03	2.4603e-05	9.4779e-08	3.6991e-10
$\eta \downarrow$	$\delta = 0.5\varepsilon$			
0.00	4.1398e-03	1.7332e-05	6.6871e-08	2.6136e-10
0.05	4.6884e-03	1.9939e-05	7.6887e-08	3.0022e-10
0.09	5.1541e-03	2.2193e-05	8.5539e-08	3.3386e-10
	Results in Kadalbajoo and Sharma [8]			
$\delta \downarrow$	$\eta = 0.5\varepsilon$			
0.00	0.09190267	0.03453494	0.01164358	0.00300463
0.05	0.10233615	0.03823132	0.01295871	0.00335137
0.09	0.11018870	0.04110846	0.01400144	0.00362925
$\eta \downarrow$	$\delta = 0.5\varepsilon$			
0.00	0.09720079	0.03640446	0.01229476	0.0031786
0.05	0.10233615	0.03823132	0.01295871	0.00335137
0.09	0.10632014	0.03965833	0.01348348	0.00349050

Example 4. Consider the problem with right end boundary layer
$\varepsilon w''(t) - w'(t) - 2w(t-\delta) + w(t) = 0$ with $w(0) = 1$, $-\delta \le t \le 0$ and $w(1) = -1$, $1 \le t \le 1+\eta$.

Table 4. Maximum errors in the solution of Example 4 for $\varepsilon = 0.1$ with different values of δ

	$N \to 8$	32	128	512
$\delta \downarrow$	Present method			
0.00	2.5312e-03	9.4444e-06	3.6657e-08	1.4317e-10
0.05	1.7367e-03	6.3970e-06	2.4904e-08	9.7252e-11
0.09	1.2459e-03	4.5388e-06	1.7815e-08	6.9587e-11
	Results in Kadalbajoo and Sharma [8]			
0.00	0.07847490	0.04678972	0.01727912	0.00443086
0.05	0.09222560	0.03828329	0.01487799	0.00380679
0.09	0.10509460	0.03149275	0.01299340	0.00331935

Example 5. Consider the differential-difference equation with right end boundary layer: $\varepsilon w''(t) - w'(t) + w(t) - 2w(t+\eta) = 0$ with $w(0) = 1$, $-\delta \le t \le 0$ and $w(1) = -1$, $1 \le t \le 1+\eta$.

Table 5. Maximum errors in the solution of Example 5 for $\varepsilon = 0.1$ with different values of η

	$N \to 8$	32	128	512
$\eta \downarrow$		Present method		
0.00	2.5312e-03	9.4444e-06	3.6657e-08	1.4317e-10
0.05	3.5567e-03	1.3428e-05	5.2345e-08	2.0442e-10
0.09	4.5639e-03	1.7380e-05	6.8290e-08	2.6661e-10
		Results in Kadalbajoo and Sharma [8]		
0.00	0.07847490	0.04678972	0.01727912	0.00443086
0.05	0.06834579	0.05516436	0.01972508	0.00506769
0.09	0.08328237	0.06168267	0.02169662	0.00558451

Example 6. Consider the equation with right end boundary layer:
$\varepsilon w''(t) - w'(t) - 2w(t-\delta) + w(t) - 2w(t+\eta) = 0$ with $w(0) = 1$, $-\delta \le t \le 0$ and $w(1) = -1$, $1 \le t \le 1+\eta$.

Table 6. Maximum errors in Example 6 for $\varepsilon = 0.1$ with different values of η and δ

	$N = 8$	$N = 32$	$N = 128$	$N = 512$
$\delta \downarrow$	$\eta = 0.5\varepsilon$		Present method	
0.00	3.7809e-03	1.4605e-05	5.7505e-08	2.2459e-10
0.05	2.8263e-03	1.0743e-05	4.2040e-08	1.6425e-10
0.09	2.1939e-03	8.2661e-06	3.2176e-08	1.2563e-10
$\eta \downarrow$	$\delta = 0.5\varepsilon$			
0.00	2.0526e-03	7.7156e-05	2.9998e-08	1.1715e-10
0.05	2.8263e-03	1.0743e-05	4.2040e-08	1.6425e-10
0.09	3.5745e-03	1.3707e-05	5.4129e-08	2.1133e-10
$\delta \downarrow$	$\eta = 0.5\varepsilon$		Results of Kadalbajoo and Sharma [8]	
0.00	0.09930002	0.03685072	0.01331683	0.00342882
0.05	0.09997296	0.03218424	0.01167102	0.00299572
0.09	0.10044578	0.02850398	0.01038902	0.00266379
$\eta \downarrow$	$\delta = 0.5\varepsilon$			
0.00	0.10055269	0.02759534	0.01007834	0.00258299
0.05	0.09997296	0.03218424	0.01167102	0.00299572
0.09	0.09944067	0.03591410	0.01297367	0.00334044

5. Discussions and Conclusion

A difference scheme using non-polynomial spline and nonstandard finite differences is implemented to solve differential-difference equation having layer behaviour. Initially, the given problem is minimized by the Taylor's expansion, to differential equation with layer structure. Then a tridiagonal difference scheme is constructed using the nonstandard finite differences in non-polynomial spline. The method is used with various examples of left layer and right layer, with distinct values of the delay parameter δ, advanced parameter η and the perturbation ε. The outcomes of the computations were compared and tabulated. The effect of the delay and the advanced parameters have been examined via graphs on the problem solutions. When the SPDDE solution exhibits the layer on the left-end, the effect of delay or advanced parameters in the layer domain is observed to be negligible, whereas in the outer region it is significant. The variation of the advanced parameter influences the solution in the same way that the change in delay has an influence but reverse effect (see Figures 1-5). In layer region as well as external region, there is an impact when the SPDDEs show right-end layer on the region with respect to the delay or advanced variations. We also observed that the layer thickness decreases as the delay parameter increases while the advanced parameter increases the layer thickness (Figures 6-8). Results show that the proposed scheme is very well suited to the exact solution.

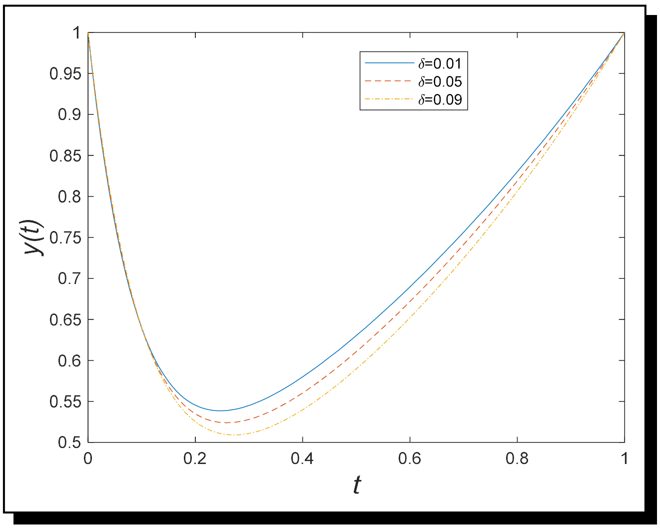

Figure 1. Numerical solution in Example 1 for $\varepsilon = 0.1$ with different values of δ

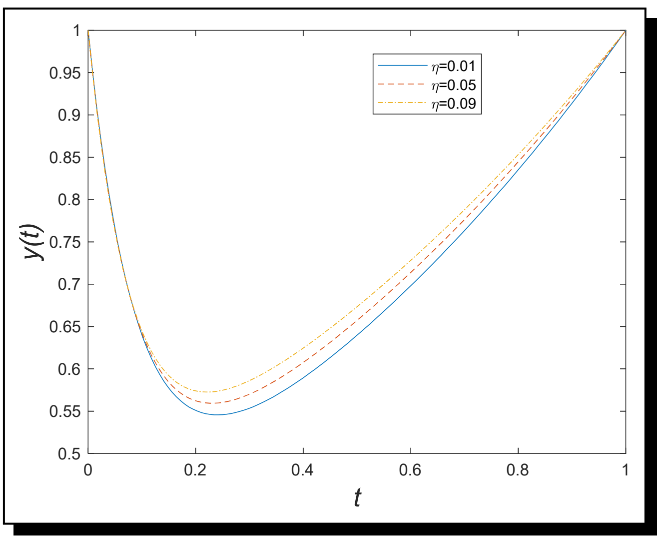

Figure 2. Numerical solution in Example 2 for $\varepsilon = 0.1$ with different values of η

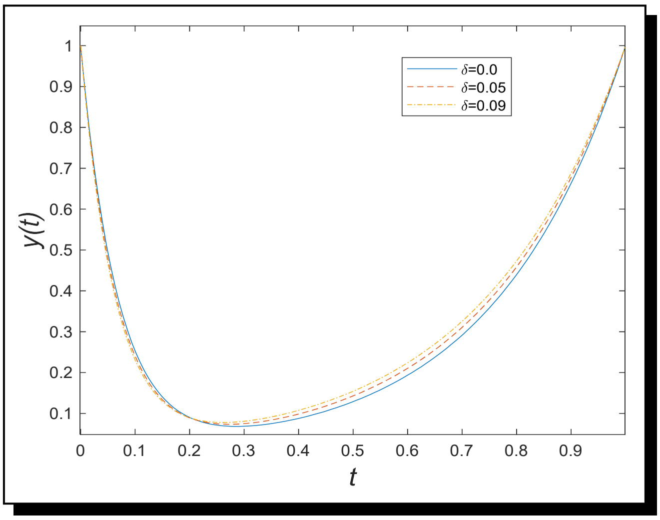

Figure 3. Numerical solution in Example 3for $\varepsilon = 0.1$ with different values of δ

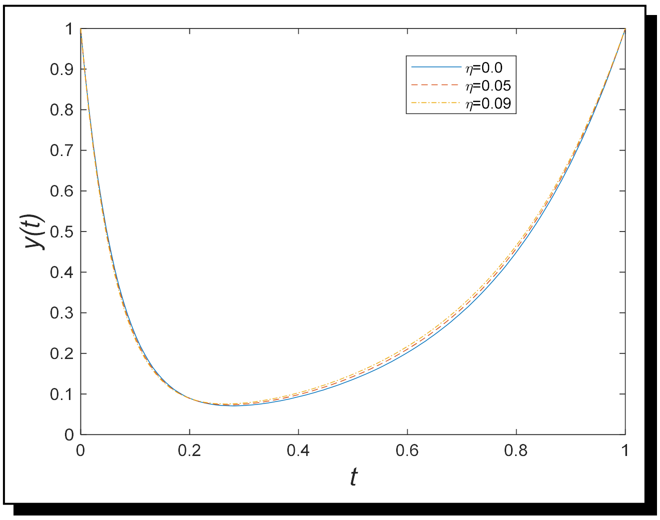

Figure 4. Numerical solution in Example 3 for $\varepsilon = 0.1$ with different values of η

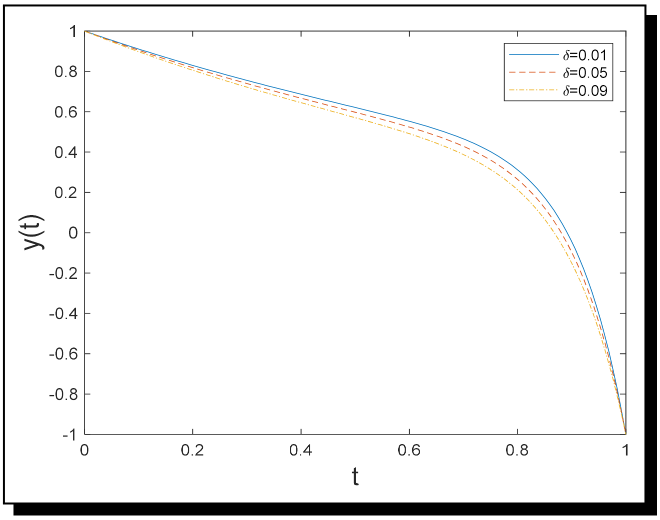

Figure 5. Numerical solution in Example 4 for $\varepsilon = 0.1$ with different values of δ

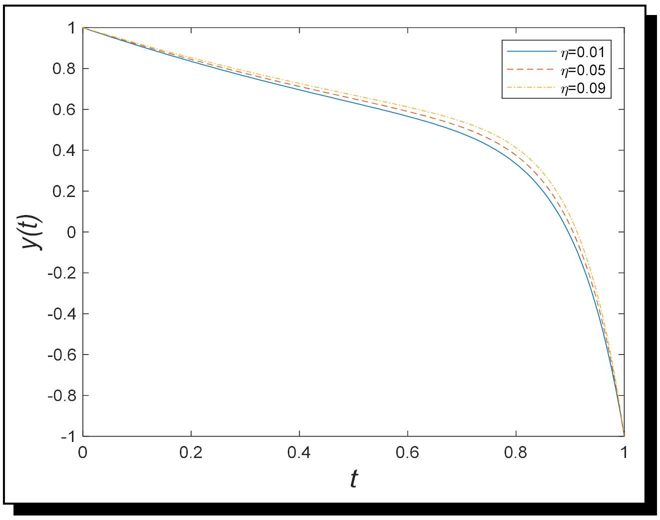

Figure 6. Numerical solution in Example 5 for $\varepsilon = 0.1$ with different values of η

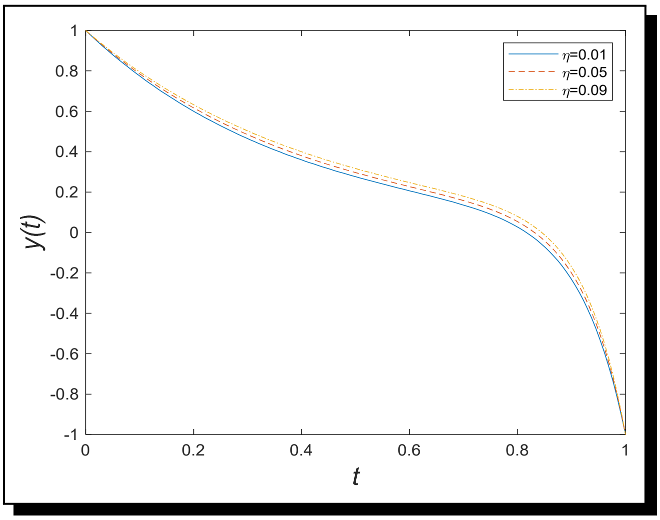

Figure 7. Numerical solution in Example 6 with $\varepsilon = 0.1$ and for different values of η

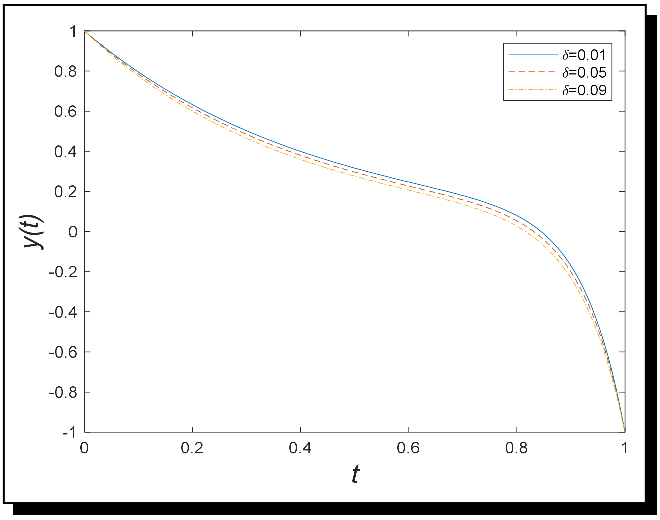

Figure 8. Numerical solution in Example 6 with $\varepsilon = 0.1$ and for different values of δ

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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