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Research Article

On Zagreb Indices of Two New Operations of Graphs

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Abstract. Recently, Wang *et al*. (2017) introduced two new operations of graphs. In this paper we establish explicit expressions of some Zagreb indices viz. first Zagreb index, second Zagreb index and forgotten topological index of these two newly proposed operations of graphs. Then as an application we further establish explicit formulae of some other topological indices of the two operations of graphs.

Keywords. Degree of vertex; Zagreb indices; Operations of graphs

MSC. 05C76; 05C07

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1. Introduction

Throughout the paper we consider only simple graphs. $V(G)$ and $E(G)$ are respectively the set of vertices and set of edges of a graph G. The degree of a vertex $u \in V(G)$ is denoted by $d_G(u)$; if their is no confusion we simply write it as $d(u)$. Two vertices u and v are called adjacent if there is an edge connecting them. The connecting edge is usually denoted by uv. Any unexplained graph theoretic symbols and definitions may be found in [17].

Topological indices are the numerical values which are associated with a graph structure. These graph invariants are utilized for modeling information of molecules in structural chemistry and biology. Over the years many topological indices have been proposed and studied based on degree, distance and other parameters of graph. Some of them may be found in [4, 9]. Historically Zagreb indices can be considered as the first degree-based topological indices, which came into picture during the study of total π-electron energy of alternant hydrocarbons by Gutman and Trinajstić in 1972 [11]. But these indices are recognized as topological indices much later (almost after 30 years, due to their completely different purpose of utility). Since these indices were coined, various studies related to different aspects of these indices are reported, for detail see the papers [3, 6, 10, 14, 18] and the references therein.

The first and second Zagreb indices of a graph G are defined as

$$M_1(G) = \sum_{u\in V(G)} d_G^2(u) = \sum_{uv\in E(G)} (d_G(u)+d_G(v)),$$

$$M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v).$$

Another Zagreb index which was reintroduced as 'forgotten topological index' by Furtula and Gutman in [7] can be defined as

$$M_3(G) = \sum_{u\in V(G)} d_G^3(u) = \sum_{uv\in E(G)} (d_G^2(u)+d_G^2(v)).$$

The hyper Zagreb index, which was put forward by Shirdel *et al*. [15] can be defined as

$$HM(G) = \sum_{uv\in E(G)} (d_G(u)+d_G(v))^2.$$

In 2014, Furtula *et al*. [8], propose another topological index during their study on difference of Zagreb indices. They name this index as 'Reduced second Zagreb index', which can be defined as

$$RM_2(G) = \sum_{uv\in E(G)} (d_G(u)-1)(d_G(v)-1).$$

It is also interesting to study the graph invariants which take into account the similar contributions of non-adjacent pairs of vertices. Such graph invariants are known as "Coindices". The first Zagreb coindex put forward by Dǒslić [5] can be defined as

$$\overline{M_1}(G) = \sum_{uv\notin E(G)} (d_G(u)+d_G(v)).$$

Analogously, they also define the second Zagreb coindex as

$$\overline{M_2}(G) = \sum_{uv\notin E(G)} d_G(u)d_G(v).$$

The forgotten topological coindex or F-coindex [2] is defined as

$$\overline{M_3}(G) = \sum_{uv\notin E(G)} (d_G^2(u)+d_G^2(v)).$$

Graph operations play a very important role in chemical graph theory, as some chemically interesting graphs can be obtained by different graph operations on some general or particular graphs. Hence it is also important to compute an index of various operations of graphs, for obvious reason that these results can be very helpful in determining the value of that index for complex graph structures.

In this paper we consider two new operations of graphs proposed by Wang *et al*. in [16] and establish explicit expressions for the Zagreb indices of these graph products in terms of the topological indices of the participating graphs. The rest of the paper is organized as follows. In Section 2 we reproduce the two graph operations under consideration. In Section 3 main results are presented. In Section 4, we apply the results to establish further expressions for some other topological indices.

2. The Two New Operations of Graphs

The Cartesian product of graphs G and H, denoted by $G\square H$, is the graph with vertex set $V(G)\square V(H)=\{(a,v): a\in V(G), v\in V(H)\}$, and (a,v) is adjacent to (b,w), whenever $a=b$ and $(v,w)\in E(H)$, or $v=w$ and $(a,b)\in E(G)$. More detail on Cartesian product and some other operations of graphs may be found in [13]. In 2017, Wang *et al*. [16] proposed the following two operations of graphs and also studied their adjacency spectrum. We reproduce the figure in [16] to make the discussion self expository.

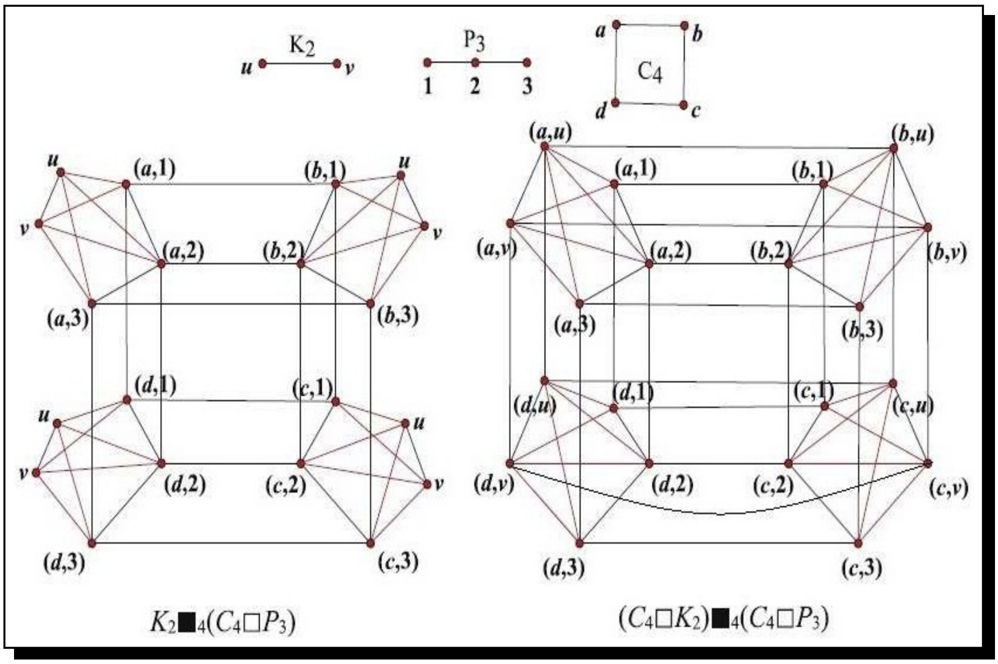

Figure 1. Two new operations [16]

Definition 2.1. Let $G_{1i}=G_1$ and $G_{2i}=G_2$ $(1\le i\le k)$ be k copies of graphs G_1 and G_2, respectively, G_j $(j=3,4)$ is an arbitrary graph.

- The first operation $G_1\blacksquare_k(G_3\square G_2)$ of G_1, G_2 and G_3 is obtained by making the Cartesian product of two graphs G_3 and G_2, thus produces k copies G_{2i} $(1\le i\le k)$ of G_2, then makes k joins $G_{1i}\vee G_{2i}$, $i=1,2,\dots,k$.
- The second operation $(G_4\square G_1)\blacksquare_k(G_3\square G_2)$ of G_1, G_2, G_3 and G_4 is obtained by making the Cartesian product of two graphs G_3 and G_2, produces k copies G_{2i} $(1\le i\le k)$ of G_2 and making the Cartesian product of two graphs G_4 and G_1, produces k copies G_{1i} $(1\le i\le k)$ of G_1, then makes k joins $G_{1i}\vee G_{2i}$, $i=1,2,\dots,k$.

For an example, we consider $G_1=K_2$, $G_2=P_2$ and $G_3=G_4=C_4$ and hence obtain the graphs $K_2\blacksquare_4(C_4\square P_3)$ and $(C_4\times K_2)\blacksquare_4(C_4\square P_3)$, which are shown in Figure 1. It is clear from the definitions of the two operations that $|E(G_1\blacksquare_k(G_3\square G_2))| = k(m_1+m_2+n_1n_2)+n_2m_3$, $|E((G_4\square G_1)\blacksquare_k(G_3\square G_2))| = k(m_1+n_1n_2+m_2)+n_1m_4+m_3n_2$ and $|V(G_1\blacksquare_k(G_3\square G_2))| = |V((G_4\square G_1)\blacksquare_k(G_3\square G_2))| = k(n_1+n_2)$. Also, it is to be exclusively mentioned that $|V(G_3)| = |V(G_4)| = k$ but $G_3 \neq G_4$ in general.

3. Main Results

Let us start the discussion with the following known theorems:

Lemma 3.1 ([14]). *Let $G_1,G_2,\dots,G_n$ be graphs with $V_i = V(G_i)$ and $E_i = E(G_i)$, $1 \le i \le n$, and $V = V(\square_{i=1}^n G_i)$. Then* $M_1(\square_{i=1}^n G_i) = |V|\sum_{i=1}^n \frac{M_1(G_i)}{|V_i|} + 4|V|\sum_{i\neq j,j=1}^n \frac{|E_i||E_j|}{|V_i||V_j|}$.

Lemma 3.2 ([14]). *Let $G_1,G_2,\dots,G_n$ be graphs with $V_i = V(G_i)$ and $E_i = E(G_i)$, $1 \le i \le n$, and $V = V(\square_{i=1}^n G_i)$ and $E = E(\square_{i=1}^n G_i)$. Then* $M_2(\square_{i=1}^n G_i) = |V|\sum_{i=1}^n\left(\frac{M_2(G_i)}{|V_i|} + 3M_1(G_i)\left(\frac{|E|}{|V_i|} - \frac{|V||E_i|}{|V_i|^2}\right)\right) + 4|V|\sum_{i,j,k=1 i\neq j,i\neq k,j\neq k}^n \frac{|E_i||E_j||E_k|}{|V_i||V_j||V_k|}$.

Lemma 3.3 ([1]). *Let $G_1,G_2,\dots,G_n$ be graphs with $V_i = V(G_i)$ and $E_i = E(G_i)$, $1 \le i \le n$, and $V = V(\square_{i=1}^n G_i)$ and $E = E(\square_{i=1}^n G_i)$. Then* $M_3(\square_{i=1}^n G_i) = |V|\sum_{i=1}^n \frac{F(G_i)}{|V_i|} + 6|V|\sum_{i,j=1,i\neq j}^n \frac{M_1(G_i)}{|V_i|}\cdot\frac{|E_j|}{V_j} + 8|V|\sum_{p,q,r=1,p\neq q\neq r}^n \frac{|E_p||E_q||E_r|}{|V_p||V_q||V_r|}$.

Now, we first propose the following lemma which can easily be proved from Definition 2.1 of the two graph operations and then we present our main results:

Lemma 3.4. *Let G_1,G_2,G_3 and G_4 be four graphs with $|V(G_i)| = n_i$, $|E(G_i)| = m_i$ where $i = 1,2,3,4$. Then,*

$$d_{(G_1\blacksquare_k(G_3\square G_2))}(u) = \begin{cases} d_{G_1}(u)+n_2 & \text{if } u \in V(G_1) \\ d_{G_3}(u_3)+d_{G_2}(u_2)+n_1 & \text{if } u=(u_3,u_2)\in V(G_3\square G_2) \end{cases}$$

and

$$d_{((G_4\square G_1)\blacksquare_k(G_3\square G_2))}(u) = \begin{cases} d_{G_4}(u_4)+d_{G_1}(u_1)+n_2 & \text{if } u=(u_4,u_1)\in V(G_4\square G_1) \\ d_{G_3}(u_3)+d_{G_2}(u_2)+n_1 & \text{if } u=(u_3,u_2)\in V(G_3\square G_2). \end{cases}$$

3.1 Zagreb Indices of $G_1\blacksquare_k(G_3\square G_2)$

Theorem 3.5. *Let G_1, G_2 and G_3 be graphs with $|V_i| = |V(G_i)| = n_i$, $|E_i| = |E(G_i)| = m_i$, $1 \le i \le 3$ and $n_3 = k$. Then*

$$\begin{aligned} M_1(G_1\blacksquare_k(G_3\square G_2)) &= k(M_1(G_1)+M_1(G_2)) + n_2M_1(G_3) + 4k(m_1n_2+n_1m_2) \\ &\quad + kn_1n_2(n_1+n_2) + 4m_3(n_1n_2+2m_2). \end{aligned}$$

Proof. We divide the set of vertices into two categories, where $u \in V(G_1)$ or $u = (u_3,u_2) \in V(G_3\square G_2)$. Then

$$M_1(G_1\blacksquare_k(G_3\square G_2)) = \sum_{u\in V(G_1\blacksquare_k(G_3\square G_2))} d^2_{G_1\blacksquare_k(G_3\square G_2)}(u)$$

$$= k \sum_{u \in V(G_1)} d^2_{G_1 \blacksquare_k (G_3 \square G_2)}(u) + \sum_{u=(u_3,u_2) \in V(G_3 \square G_2)} d^2_{G_1 \blacksquare_k (G_3 \square G_2)}(u)$$

$$= k \sum_{u \in V(G_1)} (d_{G_1}(u) + n_2)^2 + \sum_{(u_3,u_2) \in V(G_3 \square G_2)} (d_{G_3}(u_3) + d_{G_2}(u_2) + n_1)^2$$

$$= k \sum_{u \in V(G_1)} (d^2_{G_1}(u) + 2n_2 d_{G_1}(u) + n_2^2) + \sum_{u_3 \in V(G_3)} \sum_{u_2 \in V(G_2)} (d^2_{G_3}(u_3) + d^2_{G_2}(u_2)$$

$$+ n_1^2 + 2n_1 d_{G_3}(u_3) + 2n_1 d_{G_2}(u_2) + 2d_{G_3}(u_3) d_{G_2}(u_2))$$

$$= kM_1(G_1) + 2n_2 k 2m_1 + kn^2 n_1 + n_2 M_1(G_3) + n_3 M_1(G_2) + n_1^2 n_2 n_3$$

$$+ 2n_1 n_3 2m_2 + 2n_1 n_2 2m_3 + 2(2m_3)2m_2$$

$$= k(M_1(G_1) + M_1(G_2)) + n_2 M_1(G_3) + 4k(m_1 n_2 + n_1 m_2)$$

$$+ kn_1 n_2(n_1 + n_2) + 4m_3(n_1 n_2 + 2m_2).$$

Hence the result. □

Theorem 3.6. *Let G_1, G_2 and G_3 be graphs with $|V(G_i)| = n_i$, $|E(G_i)| = m_i$, where $i = 1,2,3$ and $n_3 = k$. Then,*

$$M_2(G_1 \blacksquare_k (G_3 \square G_2)) = kn_2 M_1(G_1) + (3m_3 + n_1 k) M_1(G_2) + (3m_2 + n_1 n_2) M_1(G_3)$$
$$+ kM_2(G_1) + kM_2(G_2) + n_2 M_2(G_3) + k(n_2)^2 m_1 + 8n_1 m_2 m_3$$
$$+ (n_1)^2 n_2 m_3 + (n_1)^2 k m_2 + (n_2 m_3 + k m_2)(4m_1 + 2n_1 n_2)$$
$$+ k((n_1)^2 (n_2)^2 + 2m_1 n_1 n_2).$$

Proof. In $M_2(G_1 \blacksquare_k (G_3 \square G_2)) = \sum_{uv \in E(G_1 \blacksquare_k (G_3 \square G_2))} d(u)d(v)$, the edges are classified into three categories.

Case 1: If $u, v \in V(G_1)$ and $uv \in E(G_1)$, then

$$\sum_{uv \in E(G_1 \blacksquare_k (G_3 \square G_2))} d(u)d(v) = k \sum_{uv \in E(G_1)} (d_{G_1}(u) + n_2)(d_{G_1}(v) + n_2)$$
$$= k \left[\sum_{uv \in E(G_1)} (d_{G_1}(u) d_{G_1}(v)) + n_2 \sum_{uv \in E(G_1)} (d_{G_1}(u) + d_{G_1}(v)) + n_2^2 m_1 \right]$$
$$= k \left[M_2(G_1) + n_2 M_1(G_1) + n_2^2 m_1 \right]. \tag{3.1}$$

Case 2: If $uv \in E(G_3 \square G_2)$, such that $u = (u_3, u_2)$ and $v = (v_3, v_2)$. Then

$$\sum_{uv \in E(G_1 \blacksquare_k (G_3 \square G_2))} d(u)d(v) = \sum_{uv \in E(G_3 \square G_2)} (d_{G_3}(u_3) + d_{G_2}(u_2) + n_1)(d_{G_3}(v_3) + d_{G_2}(v_2) + n_1)$$
$$= \left(\sum_{uv \in E(G_3 \square G_2)} d_{G_3 \square G_2}(u) \cdot d_{G_3 \square G_2}(v) + n_1 \right)$$
$$\cdot \sum_{uv \in E(G_3 \square G_2)} (d_{G_3 \times G_2}(u) + d_{G_3 \times G_2}(v)) + n_1^2 |E(G_3 \square G_2)|$$
$$= (M_2(G_3 \square G_2) + n_1) M_1(G_3 \square G_2) + n_1^2 (n_2 m_3 + n_3 m_2).$$

Using the results from Lemma 3.1 and 3.2, we have

$$\begin{aligned}\sum_{uv\in E(G_1\blacksquare_k(G_3\Box G_2))} d(u)d(v) &= n_3M_2(G_2)+n_2M_2(G_3)+3m_2M_1(G_3)+3m_3M_1(G_2)\\ &\quad +n_1\left[n_3M_1(G_2)+n_2M_1(G_3)+8m_2m_3\right]+n_1^2(n_2m_3+n_3m_2)\\ &= kM_2(G_2)+n_2M_2(G_3)+(3m_3+n_1k)M_1(G_2)\\ &\quad +(3m_2+n_1n_2)M_1(G_3)+8n_1m_2m_3+n_1^2n_2m_3+n_1^2km_2. \qquad (3.2)\end{aligned}$$

Case 3: If $uv\in E(G_1\blacksquare_k(G_3\Box G_2))$, such that $u\in V(G_1)$ and $v=(v_3,v_2)\in V(G_3\Box G_2)$. Then

$$\begin{aligned}\sum_{uv\in E(G_1\blacksquare_k(G_3\Box G_2))} d(u)d(v) &= \sum_{uv\in E(G_1\blacksquare_k(G_3\Box G_2))} (d_{G_1}(u)+n_2)(d_{G_3}(v_3)+d_{G_2}(v_2)+n_1)\\ &= \sum_{u\in V(G_1)}\sum_{(v_3,v_2)\in V(G_3\Box G_2)} \left[d_{G_1}(u)(d_{G_3}(v_3)+d_{G_2}(v_2))+n_1d_{G_1}(u)\right.\\ &\quad \left.+n_1n_2+n_2(d_{G_3}(v_3)+d_{G_2}(v_2))\right]\\ &= \sum_{u\in V(G_1)} d_{G_1}(u)\sum_{v\in V(G_3\Box G_2)} d_{G_3\Box G_2}(v)\\ &\quad +\sum_{u\in V(G_1)} d_{G_1}(u)\sum_{v\in V(G_3\Box G_2)} n_1+n_1n_2(n_1.n_3n_2)\\ &\quad +n_2\sum_{u\in V(G_1)}\sum_{v\in V(G_3\Box G_2)} d_{G_3\Box G_2}(v)\\ &= 2m_1.2|E(G_3\Box G_2)|+2m_1n_1|V(G_3\Box G_2)|+n_3n_1^2n_2^2\\ &\quad +n_1n_2.2|E(G_3\Box G_2)|\\ &= 4m_1(n_2m_3+n_3m_2)+2m_1n_1n_2m_3+n_3n_1^2n_2^2\\ &\quad +2n_1n_2(n_3m_2+n_2m_3)\\ &= (n_2m_3+km_2)(4m_1+2n_1n_2)+k(n_1^2n_2^2+2m_1n_1n_2). \qquad (3.3)\end{aligned}$$

Combining the expressions (3.1), (3.2) and (3.3), we have the result. □

Theorem 3.7. *Let G_1,G_2 and G_3 be any graphs with $|V(G_i)|=n_i$, $|E(G_i)|=m_i$, where $i=1,2,3$ and $n_3=k$. Then,*

$$\begin{aligned}M_3(G_1\blacksquare_k(G_3\Box G_2)) &= k(M_3(G_1)+kM_3(G_2)+n_2M_3(G_3)+3n_2M_1(G_1))\\ &\quad +(6m_3+3n_1k)M_1(G_2)+(6m_2+3n_1n_2)M_1(G_3)\\ &\quad +n_2(n_1n_2^2+6n_2m_1+n_1^3k+6n_1^2m_3)+3n_1(2n_1km_2+8m_2m_3).\end{aligned}$$

Proof.

$$\begin{aligned}M_3(G_1\blacksquare_k(G_3\Box G_2)) &= \sum_{u\in V(G_1\blacksquare_k(G_3\Box G_2))} d^3(u)\\ &= k\sum_{u\in V(G_1)}(d_{G_1}(u)+n_2)^3+\sum_{u=(u_3,u_2)\in V(G_3\Box G_2)}(d_{G_3}(u_3)+d_{G_2}(u_2)+n_1)^3\\ &= k\sum_{u\in V(G_1)}\left[d_{G_1}^3(u))+n_2^3+3n_2^2d_{G_1}(u)+3n_2d_{G_1}^2(u)\right]\\ &\quad +\sum_{u\in V(G_3\Box G_2)}\left[d_{G_3\Box G_2}^3(u)+n_1^3+3n_1^2d_{G_3\Box G_2}(u)+3n_1d_{G_3\Box G_2}^2(u)\right]\end{aligned}$$

$$= k(M_3(G_1) + n_2^3 n_1 + 3n_2^2.2m_1 + 3n_2 M_1(G_1))$$
$$+ \left[M_3(G_3\Box G_2) + n_1^3 n_3 2n_2 + 3n_1^2.2(n_2m_3 + n_3m_2) + 3n_1M_1(G_3\Box G_2)\right].$$

Using the result from Lemma 3.3, we can write the above expression as

$$M_3(G_1\blacksquare_k(G_3\Box G_2)) = k(M_3(G_1) + n_2^3n_1 + 6n_2^2m_1 + 3n_2M_1(G_1)) + n_2M_3(G_3) + n_3M_3(G_2)$$
$$+ 6m_2M_1(G_3) + 6m_3M_1(G_2) + n_1^3n_3n_2 + 6n_1^2n_2m_3 + 6n_1^2n_3m_2$$
$$+ 3n_1(n_2M_1(G_3) + n_3M_1(G_2) + 8m_2m_3).$$

Now, by suitably rearranging the terms, we get the desired result. □

3.2 Zagreb Indices of $(G_4\Box G_1)\blacksquare_k(G_3\Box G_2)$

Theorem 3.8. *Let G_1, G_2, G_3 and G_4 be four graphs with $|V(G_i)| = n_i$, $|E(G_i)| = m_i$, where $i = 1,2,3,4$ and $n_3 = n_4 = k$. Then,*

$$M_1((G_4\Box G_1)\blacksquare_k(G_3\Box G_2)) = n_1(M_1(G_4) + n_2M_1(G_3)) + k(M_1(G_1) + M_1(G_2)) + n_1n_2k(n_1 + n_2)$$
$$+ 8m_1m_4 + 8m_2m_3 + 4n_1n_2(m_3 + m_4) + 4k(m_1n_2 + n_1m_2).$$

Proof.

$$M_1((G_4\Box G_1)\blacksquare_k(G_3\Box G_2)) = \sum_{u\in V(G_4\Box G_1)\blacksquare_k(G_3\Box G_2))} d^2(u)$$
$$= \sum_{(u_4,u_1)\in V(G_4\times G_1)} (d_{G_4}(u_4) + d_{G_1}(u_1) + n_2)^2$$
$$+ \sum_{(u_3,u_2)\in V(G_3\times G_2)} (d_{G_3}(u_3) + d_{G_2}(u_2) + n_1)^2$$
$$= \sum_{u_4\in V(G_4)} \sum_{u_1\in V(G_1)} ((d_{G_4}(u_4))^2 + (d_{G_1}(u_1))^2 + (n_2)^2$$
$$+ 2d_{G_4}(u_4)d_{G_1}(u_1) + 2n_2d_{G_4}(u_4) + 2n_2d_{G_1}(u_1))$$
$$+ \sum_{u_3\in V(G_3)} \sum_{u_2\in V(G_2)} ((d_{G_3}(u_3))^2 + (d_{G_2}(u_2))^2 + (n_1)^2$$
$$+ 2d_{G_3}(u_3)d_{G_2}(u_2) + 2n_1d_{G_3}(u_3) + 2n_1d_{G_2}(u_2))$$
$$= n_1M_1(G_4) + kM_1(G_1) + n_1(n_2)^2k + 2.2m_4.2m_1$$
$$+ 2n_2.2m_4.n_1 + 2n_2.2m_1k + n_2M_1(G_3) + kM_1(G_2)$$
$$+ (n_1)^2kn_2 + 2n_1.2m_3.n_2 + 2n_1.k.2m_2 + 2.2m_2.2m_3,$$

by rearranging the terms, we have the theorem. □

Theorem 3.9. *Let G_1, G_2, G_3 and G_4 be any four graphs with $|V(G_i)| = n_i$, $|E(G_i)| = m_i$ for $i = 1,2,3,4$ and also $n_3 = n_4 = k$. Then,*

$$M_2((G_4\Box G_1)\blacksquare_k(G_3\Box G_2)) = (3m_4 + n_2k)M_1(G_1) + (3m_3 + kn_1)M_1(G_2) + (3m_2 + n_1n_2)M_1(G_3)$$
$$+ (3m_1 + n_1n_2)M_1(G_4) + kM_2(G_1) + kM_2(G_2) + n_2M_2(G_3)$$
$$+ n_1M_2(G_4) + 2kn_1n_2(m_1 + m_2) + 2n_1n_2(n_1m_4 + n_2m_3)$$
$$+ km_1(n_2 + 4m_2) + 4m_2n_1(2m_3 + m_4) + n_1n_2(m_3n_1 + m_4n_2)$$

$$+4m_1(m_3n_2+2m_4n_2)+kn_1^2(m_2+n_2^2)+n_1n_2\sum_{i=i}^{k}d_{G_4}(u_4^i)d_{G_3}(v_3^i),$$

where $u_4^i \in V(G_4)$, $v_3^i \in V(G_3)$ *and for fixed* i, (u_4^i,u_1) *is adjacent to* (v_3^i,v_2) *for any* $u_1 \in V(G_1)$ *and* $v_2 \in V(G_2)$.

Proof. Here again, the edges can be classified into three categories.

Case 1: If $uv \in E(G_4\square G_1)$, let $u=(u_4,u_1)$, $v=(v_4,v_1)$. Then,

$$\begin{aligned}\sum_{uv\in((G_4\square G_1)\blacksquare_k(G_3\square G_2))} d(u)d(v) &= \sum_{uv\in E(G_4\square G_1)}(d_{G_4}(u_4)+d_{G_1}(u_1)+n_2)(d_{G_4}(v_4)+d_{G_1}(v_1)+n_2)\\
&= \sum_{uv\in E(G_4\square G_1)}(d_{G_4\square G_1}(u)+n_2)(d_{G_4\square G_1}(v)+n_2)\\
&= \sum_{uv\in E(G_4\square G_1)}(d(u)d(v)+n_2(d(u)+d(v))+n_2^2)\\
&= M_2(G_4\square G_1)+n_2M_1(G_4\square G_1)+n_2^2(n_1m_4+n_4m_1)\\
&= kM_2(G_1)+n_1M_2(G_4)+3m_4M_1(G_1)+3m_1M_1(G_4)\\
&\quad +n_2(kM_1(G_1)+n_1M_1(G_4)+8m_1m_4+n_1n_2m_4+kn_2m_1). \qquad (3.4)\end{aligned}$$

Case 2: If $uv \in E(G_3\square G_2)$, let $u=(u_3,u_2)$, $v=(v_3,v_2)$. Then similarly

$$\begin{aligned}\sum_{uv\in E((G_4\square G_1)\blacksquare_k(G_3\square g_2))} d(u)d(v) &= \sum_{uv\in E(G_3\square G_2)}(d_{G_3}(u_3)+d_{G_2}(u_2)+n_1)(d_{G_3}(v_3)+d_{G_2}(v_2)+n_1)\\
&= \sum_{uv\in E(G_3\square G_2)}(d(u)+n_1)(d(v)+n_1)\\
&= kM_2(G_2)+n_2M_2(G_3)+3m_3M_1(G_2)+3m_2M_1(G_3)\\
&\quad +n_1(kM_1(G_2)+n_2M_1(G_3)+8m_2m_3+n_1n_2m_3+kn_1m_2). \qquad (3.5)\end{aligned}$$

Case 3: Let $u \in V(G_4\square G_1)$ and $v \in V(G_3\square G_2)$ s.t. $u=(u_4^i,u_1)$ and $v=(v_3^i,v_2)$, where $V(G_3)=\{u_3^1,u_3^2,\dots,u_3^k\}$ and $V(G_4)=\{v_4^1,v_4^2,\dots,v_4^k\}$ and for fixed i, u is adjacent to v for all $u_1 \in V(G_1)$ and $v_2 \in V(G_2)$.

$$\begin{aligned}\sum_{uv\in E((G_4\square G_1)\blacksquare_k(G_3\square G_2))} d(u)d(v) &= \sum_{i=1}^{k}\sum_{u_2\in V(G_2)}\sum_{v_1\in V(G_1)}(d_{G_3}(u_3^i)+d_{G_2}(u_2)+n_1)(d_{G_4}(u_4^i)+d_{G_1}(v_1)+n_2)\\
&=4n_1m_2m_4+2n_1^2n_2m_4+4n_2m_1m_3+4m_1m_2k+2kn_1m_1n_2\\
&\quad +2n_1n_2^2m_3+2kn_1n_2m_2+kn_1^2n_2^2+n_1n_2\sum_{i=1}^{k}d_{G_4}(u_4^i)d_{G_3}(v_3^i). \qquad (3.6)\end{aligned}$$

From (3.4), (3.5) and (3.6), we get the desired result. □

Theorem 3.10. *Let* G_1, G_2, G_3 *and* G_4 *be any four graphs with* $|V(G_i)|=n_i$, $|E(G_i)|=m_i$, *where* $i=1,2,3,4$ *and* $n_3=n_4=k$. *Then,*

$$\begin{aligned}M_3((G_4\square G_1)\blacksquare_k(G_3\square G_2)) &= 3[kn_2M_1(G_1)+kn_1M_1(G_2)+n_1n_2M_1(G_3)+n_1n_2M_1(G_4)]\\
&\quad +kM_3(G_1)+kM_3(G_2)+n_2M_3(G_3)+n_1M_3(G_4)\\
&\quad +kn_1n_2(n_1^2+n_2^2)+3[n_1^2(n_2m_3+km_2)+n_2^2(km_1+m_4n_1)].\end{aligned}$$

Proof. In $M_3((G_4\Box G_1)\blacksquare_k(G_3\Box G_2))=\sum_u d^3_{(G_4\Box G_1)\blacksquare_k(G_3\Box G_2)}(u)$, we again divide the set of vertices into two categories.

Case 1: Let $u=(u_4,u_1)\in V(G_4\Box G_1)$. Here $d_{(G_4\Box G_1)\blacksquare_k(G_3\Box G_2)}(u)=d_{G_4}(u_4)+d_{G_1}(u_1)+n_2$ which can be considered as $d_{(G_4\Box G_1)\blacksquare_k(G_3\Box G_2)}(u)=d_{G_4\Box G_1}(u)+n_2$. Hence,

$$\begin{aligned}\sum_u d^3_{(G_4\Box G_1)\blacksquare_k(G_3\Box G_2)}(u)&=\sum_{u\in V(G_4\Box G_1)}(d(u)+n_2)^3\\&=\sum_{u\in V(G_4\Box G_1)}d^3(u)+n_2^3+3n_2d^2(u)+3n_2^2d(u)\\&=M_3(G_4\Box G_1)+n_2^3n_1n_4+3n_2M_1(G_4\Box G_1)+3n_2^2(n_4m_1+m_4n_1)\\&=(n_1M_3(G_4)+n_4M_3(G_1))+n_1n_2^3n_4+3n_2^2(n_4m_1+m_4n_1)\\&\quad+3n_2(n_4M_1(G_1)+n_1M_1(G_4)+8m_1m_4)\end{aligned}\tag{3.7}$$

Case 2: Let $u=(u_3,u_2)\in V(G_3\Box G_2)$. Then $d_{(G_4\Box G_1)\blacksquare_k(G_3\Box G_2)}(u)=d_{G_3}(u_3)+d_{G_2}(u_2)+n_1=d_{G_3\Box G_2}(u)+n_1$. Similarly,

$$\begin{aligned}\sum_u d^3_{(G_4\Box G_1)\blacksquare_k(G_3\Box G_2)}(u)=&(n_2M_3(G_3)+n_3M_3(G_2))+n_2n_1^3n_3+3n_1^2(n_3m_2+m_3n_2)\\&+3n_1(n_3M_1(G_2)+n_2M_1(G_3)+8m_2m_3).\end{aligned}\tag{3.8}$$

Combining (3.7) and (3.8), then putting $n_3=k=n_4$, we get the desired result. □

4. Some Applications

We will use the results obtained in the previous section and establish formulae of some more topological indices for the two operations of graphs.

Theorem 4.1. *Let G_1,G_2 and G_3 be graphs with $|V_i|=|V(G_i)|=n_i$, $|E_i|=|E(G_i)|=m_i$, $1\le i\le 3$ and $n_3=k$. Then*

$$\begin{aligned}\overline{M_1}(G_1\blacksquare_k(G_3\Box G_2))=2\beta_1-\big[&k(M_1(G_1)+M_1(G_2))+n_2M_1(G_3)+4k(m_1n_2+n_1m_2)\\&+kn_1n_2(n_1+n_2)+4m_3(n_1n_2+2m_2)\big],\end{aligned}$$

where $\beta_1=\{(km_2+n_1n_2)-(m_1+n_2m_3)\}(n-1)$.

Theorem 4.2.

$$\begin{aligned}\overline{M_1}((G_4\Box G_1)\blacksquare_k(G_3\Box G_2))=2\beta_2-\big[&n_1M_1(G_4)+n_2M_1(G_3)+k(M_1(G_1)+M_1(G_2))\\&n_1n_2k(n_1+n_2)+8m_1m_4+8m_2m_3+4n_1n_2(m_3+m_4)\\&+4k(m_1n_2+n_1m_2)\big],\end{aligned}$$

where $\beta_2=\{k(m_1+m_2+n_1n_2)+n_1m_4+m_3n_2\}(n-1)$.

Theorem 4.3. *Let G_1,G_2 and G_3 be graphs with $|V_i|=|V(G_i)|=n_i$, $|E_i|=|E(G_i)|=m_i$, $1\le i\le 3$ and $n_3=k$. Then*

$$\overline{M_2}(G_1\blacksquare_k(G_3\Box G_2))=2\gamma_1^2-\left[k\left(\frac{1}{2}+n_2\right)M_1(G_1)+\left(\frac{k}{2}+n_1k+3m_3\right)M_1(G_2)\right.$$

$$+\left(\frac{n}{2}+3m_2+n_1n_2\right)M_1(G_3)+kM_2(G_1)+kM_2(G_2)+n_2M_2(G_3)$$
$$+2k(m_1n_2+n_1m_2)+\frac{1}{2}n_1n_2(n_1+n_2)+2m_3(n_1n_2+2m_2)+kn_2^2m_1$$
$$+8n_1m_2m_3+n_1^2n_2m_3+kn_1^2m_2+(n_2m_3+km_2)(4m_1+2n_1n_2)$$
$$+k(n_1^2n_2^2+2m_1n_1n_2)\Big],$$

where $\gamma_1=k(m_2+n_1n_2)+(m_1+n_2m_3)$.

Theorem 4.4. *Let* G_1,G_2,G_3 *and* G_4 *be four graphs with* $|V(G_i)|=n_i$, $|E(G_i)|=m_i$ *where* $i=1,2,3,4$ *and* $n_3=n_4=k$. *Then,*

$$\overline{M_2}((G_4\square G_1)\blacksquare_k(G_3\square G_2))=2\gamma_2^2-\Big[\left(\frac{k}{2}+kn_2\right)M_1(G_1)+\left(\frac{k}{2}+3m_3+kn_1\right)M_1(G_2)$$
$$+(n_2+3m_2+n_1n_2)M_1(G_3)+(\frac{n_1}{2}+3m_1+3m_4+n_1n_2)M_1(G_4)$$
$$+kM_2(G_1)+kM_2(G_2)+n_2M_2(G_3)+n_1M_2(G_4)$$
$$+kn_1n_2(m_1+m_2+n_1+n_2)+8m_1m_4+8m_2m_3$$
$$+n_1(2n_1n_2m_4+2n_1n_2m_2+4n_2m_3$$
$$+4n_2m_4+n_2m_3n_1+n_2^2m_4)+4k(m_1n_2+n_1m_2)$$
$$+4m_1(m_3n_2+2m_4n_2)+kn_1^2(m_2^2+n_2^2)$$
$$+n_1n_2\sum_{i=1}^{k}d_{G_4}(u_4^i)d_{G_3}(v_3^i)\Big],$$

where $\gamma_2=k(m_1+m_2+n_1n_2)+n_1m_4+m_3n_2$. *Also,* $u_4^i\in V(G_4)$, $v_3^i\in V(G_3)$ *and for fixed* i, (u_4^i,u_1) *is adjacent to* (v_3^i,v_2) *for any* $u_1\in V(G_1)$ *and* $v_2\in V(G_2)$.

Theorem 4.5. *Let* G_1,G_2 *and* G_3 *be graphs with* $|V_i|=|V(G_i)|=n_i$, $|E_i|=|E(G_i)|=m_i$, $1\le i\le 3$ *and* $n_3=k$. *Then*

$$\overline{M_3}(G_1\blacksquare_k(G_3\square G_2))=k(\alpha_1-3n_2)M_1(G_1)+(\alpha_1k-6m_3-3n_1)M_1(G_2)$$
$$+(\alpha_1n_2-6m_2-3n_1n_2)M_1(G_3)-kM_3(G_1)-k^2M_3(G_2)$$
$$-n_2kM_3(G_3)+4\alpha_1k(m_1n_2+n_1m_2)+\alpha_1kn_1n_2(n_1+n_2)$$
$$+4\alpha_1m_3(n_1n_2+2m_2)-n_2(n_1n_2^2+6n_2m_1+kn_1^3+6n_1^2m_3)$$
$$-3n_1(2kn_1m_2+8m_2m_3),$$

where $\alpha_1=k(n_1+n_2)-1$.

Theorem 4.6. *Let* G_1,G_2,G_3 *and* G_4 *be four graphs with* $|V(G_i)|=n_i$, $|E(G_i)|=m_i$, *where* $i=1,2,3,4$ *and* $n_3=n_4=k$. *Then,*

$$\overline{M_3}((G_4\square G_1)\blacksquare_k(G_3\square G_2))=(\alpha_2k-3kn_2)M_1(G_1)+(\alpha_2k-3kn_1)M_1(G_2)$$
$$+(\alpha_2n_2-n_1n_2)M_1(G_3)+(\alpha_2n_1-n_1n_2)M_1(G_4)-kM_3(G_1)$$
$$-kM_3(G_2)-n_2M_3(G_3)-n_1M_3(G_4)$$

$$+kn_1n_2(\alpha_2n_1+\alpha_2n_2-n_1^2-n_2^2)$$
$$+8\alpha_2m_1m_4+8\alpha_2m_2m_3+4\alpha_2n_1n_2(m_3+m_4)$$
$$+4\alpha_2k(m_1n_2+n_1m_2)-3\left[n_1^2(n_2m_3+km_2)+n_2^2(km_1+m_4n_1)\right],$$

where $\alpha_2=k(n_1+n_2)-1$.

Theorem 4.1 to Theorem 4.6 can be proved using the results obtained in Theorem 3.5 to Theorem 3.10 and also using the relations between the topological indices and coindices in [2, 12].

Theorem 4.7. *Let* G_1, G_2 *and* G_3 *be graphs with* $|V_i|=|V(G_i)|=n_i$, $|E_i|=|E(G_i)|=m_i$, $1\le i\le 3$ *and* $n_3=k$. *Then*

$$\begin{aligned}HM(G_1\blacksquare_k(G_3\square G_2)) =&5kn_2M_1(G_1)+(12m_3+5n_1k)M_1(G_2)+(12m_2+5n_1n_2)M_1(G_3)\\&+2kM_2(G_1)+2kM_2(G_2)+2n_2M_2(G_3)+kM_3(G_1)+k^2M_3(G_2)\\&+kn_2M_3(G_3)+16n_1m_2m_3+2n_1^2n_2m_3+2kn_1^2m_2+2(n_2m_3+km_2)\\&(4m_1+2n_1n_2)+2k(n_1^2n_2^2+2m_1n_1n_2)+n_2(n_1n_2^2+6n_2m_1+kn_1^3\\&+6n_1^2m_3)+3m(2kn_1m_2+8m_2m_3).\end{aligned}$$

Theorem 4.8. *Let* G_1, G_2, G_3 *and* G_4 *be four graphs with* $|V(G_i)|=n_i$, $|E(G_i)|=m_i$ *where* $i=1,2,3,4$ *and* $n_3=n_4=k$. *Then,*

$$\begin{aligned}HM((G_4\square G_1)\blacksquare_k(G_3\square G_2)) =&5kn_2M_1(G_1)+(5kn_1+6m_3)M_1(G_2)+(5n_1n_2+6m_2)M_1(G_3)\\&+(3n_1n_2+6m_4+6m_1+2n_1)M_1(G_4)+2kM_2(G_1)+2kM_2(G_2)\\&+2n_2M_2(G_3)+3kM_3(G_1)+3kM_3(G_2)+3n_2M_3(G_3)+3n_1M_3(G_4)\\&+kn_1n_2(n_1^2+n_2^2)+3\left[n_1^2(n_2m_3+km_2)+n_2^2(km_1+m_4n_1)\right]\\&+2n_2(8m_1m_4+n_1n_2m_4+km_1)+8n_1m_2m_3+2n_1^2n_2m_3+2kn_1^2m_2.\end{aligned}$$

Theorem 4.7 and 4.8 can be proved from the fact that for any graph G, $HM(G)=M_3(G)+2M_2(G)$ and then using the results obtained in Section 3.

Theorem 4.9. *Let* G_1, G_2 *and* G_3 *be graphs with* $|V_i|=|V(G_i)|=n_i$, $|E_i|=|E(G_i)|=m_i$, $1\le i\le 3$ *and* $n_3=k$. *Then*

$$\begin{aligned}RM_2(G_1\blacksquare_k(G_3\square G_2)) =&k(n_2-1)M_1(G_1)+(3m_3+n_1k-k)M_1(G_2)+(3m_2+n_1n_2\\&-kn_2)M_1(G_3)+kM_2(G_1)+(k+n_2)M_2(G_2)+kn_2^2m_1+8n_1m_2m_3\\&+n_1^2n_2m_3+n_1^2m_2k+(n_2m_3+km_2+4m_1+2n_1n_2)+k(n_1^2n_2^2\\&+2m_1n_1n_2)-4k(m_1n_2+n_1m_2)-kn_1n_2(n_1+n_2)-4m_3(n_1n_2+2m_2)\\&+k(m_1+m_2+n_1n_2)+n_2m_3.\end{aligned}$$

Theorem 4.10. *Let G_1,G_2,G_3 and G_4 be four graphs with $|V(G_i)| = n_i$, $|E(G_i)| = m_i$, where $i = 1,2,3,4$ and $n_3 = n_4 = k$. Then,*

$$\begin{aligned}RM_2((G_4\square G_1)\blacksquare_k(G_3\square G_2)) =& k(n_2-1)M_1(G_1)+(3m_3+kn_1-k)M_1(G_2)\\&+3m_2M_1(G_3)+(3m_1+3m_4+n_1n_2-n_1)M_1(G_4)+kM_2(G_2)\\&+n_2M_2(G_3)+n_1M_2(G_4)+2kn_1n_2(m_1+m_2)+2n_1n_2(n_1m_4\\&+n_2m_2)+km_1(n_2+km_2)+4m_2n_1(2m_3+m_4)+n_1n_2(m_3n_1\\&+m_4n_2)+4m_1(m_3n_2+2m_4n_2)+kn_1^2(m_2+n_2^2)\\&+n_1n_2\sum_{i=1}^{k} d_{G_4}(u_4^i)d_{G_3}(v_3^i)-kn_1n_2(n_1+n_2)-8m_1m_4-8m_2m_3\\&-4n_1n_2(m_3+m_4)-4k(m_1n_2+m_2n_1)+k(m_1+m_2+n_1n_2)\\&+(n_1m_4+n_2m_3),\end{aligned}$$

where $u_4^i \in V(G_4)$, $v_3^i \in V(G_3)$ and for fixed i, (u_4^i,u_1) is adjacent to (v_3^i,v_2) for any $u_1 \in V(G_1)$ and $v_2 \in V(G_2)$.

Theorem 4.9 and 4.10 can be proved using the results obtained in Section 3 and also from the fact that for any graph G, $RM_2(G) = M_2(G) - M_1(G) + |E(G)|$.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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