Communications in Mathematics and Applications
Vol. 10, No. 3, pp. 361–368, 2019
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v10i3.1275

Research Article

Weighted (k,n)-arcs of Type ($n-q$,n) and Maximum Size of (h,m)-arcs in PG(2,q)

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Abstract. In this paper, we introduce a generalized weighted (k,n)-arc of two types in the projective plane of order q, where q is an odd prime number. The sided result of this work is finding the largest size of a complete (h,m)-arcs in PG$(2,q)$, where h represents a point of weight zero of a weighted (k,n)-arc. Also, we prove that a $\left(\frac{q(q-1)}{2}+1,\frac{q+1}{2}\right)$-arc is a maximal arc in PG$(2,q)$.

Keywords. (k,n)-arcs; Weighted (k,n)-arc; PG$(2,q)$; PG$(2,\text{prime})$; Projective plane; Galois plane; Algebraic geometry

MSC. 51E21; 14N99; 51A05

Received: July 15, 2019 **Accepted:** September 13, 2019

1. Introduction

The concept of weighted (k,n)-arcs was originally established by Tallini-Scafati [10] in 1971. In order nine Galois plane, Wilson [11] in 1986 mentioned that there is a $(88,14,f)$-arc of class $(11,14)$. In addition, a $(10,7,f)$-arc of type $(4,7)$ in PG$(2,3)$ was proved by Wilson. In 1989, Hameed [4] studied the existence and non-existence of weighted (k,n)-arcs in PG$(2,9)$ as well as he proved that there exist a $(81,12,f)$-arc of type $(9,12)$ and a $(85,13,f)$-arc of type $(10,13)$. Hill and Love [6] in 2003 discussed the $(22,4)$-arcs in PG$(2,7)$. They discussed the optimal linear codes and arcs in projective geometries. In 2012, Hamilton [5] constructed a new maximal arcs in PG$(2,2^h)$, $h \geq 5$, h odd. In 1999, Marcugini, Milani and Pambianco [8] were able to compute the maximum size of $(n,3)$-arcs in PG$(2,11)$. A detailed work of $(k,3)$-arcs in PG$(2,q)$, with $q \leq 13$ was investigated by Coolsaet and Sticker [1] in 2012.

To facilitate the idea of the weighted (k,n)-arcs, we list in the preliminaries section some significant definitions and corollaries. Furthermore, important theorems and related lemmas with their proofs are given in the same section. Finally, a new maximal arc in a projective plane of order q is provided and proved.

2. Preliminaries

Definition 2.1 ([7]). Let GF$(p) = \mathbb{Z}/p\mathbb{Z}$, where p is a prime number, and suppose that $f(x)$ is a polynomial of degree σ over GF(p), and $f(x)$ is irreducible, then

$$\mathrm{GF}(q) = \mathrm{GF}(p^\sigma) = \mathrm{GF}(p)[x]/f(x) = \{a_0 + a_1 t + \cdots + a_{\sigma-1} t^{\sigma-1} : a_i \text{ in } \mathrm{GF}(P),\ f(t) = 0\}.$$

Definition 2.2 ([7]). A projective plane over GF(q) is a projective space that is two-dimensional and denoted by PG$(2,q)$ or π which contains q^2+q+1 lines, every line contains $q+1$ points that satisfy the following axioms:

(i) Any two distinct points determine a unique line;

(ii) Any two distinct lines intersect in exactly one point;

(iii) There exist four distinct points such that no three of them are on a same line.

Definition 2.3 ([4]). A t_n-arc can be defined as a set of t_n points such that there is no three points are lying on the same line.

Lemma 2.4 ([4]). *Let $t(p)$ represents the number of all tangents through p of t_n-arc, and suppose that T_i represents the number of all i-secants of t_n in* PG$(2,q)$*, then*

(i) $t(p) = q+2-t_n$;

(ii) $T_2 = (t_n(t_n-1))/2$;

(iii) $T_1 = t_n t,\ t = q+2-t_n$;

(iv) $T_0 = q(q-1)/2 + t(t-1)/2$;

(v) $T_0 + T_1 + T_2 = q^2+q+1$.

Definition 2.5 ([4]). The set of t_n lines such that no three are concurrent is called a dual of t_n-arc.

Lemma 2.6. *Let $t(l)$ be the number of points lies on l and let S_i be the number of points which pass through it $i2$-secant, then*

(i) $t(l) = q+2-t_n$;

(ii) $S_2 = (t_n(t_n-1))/2$;

(iii) $S_1 = t_n t,\ t = q+2-t_n$;

(iv) $S_0 = q(q-1)/2 + t(t-1)/2$;

(v) $S_0 + S_1 + S_2 = q^2+q+1$.

Definition 2.7 ([7]). A (h,m)-arc $\mathcal{H}$ is a set of h points such that there are m but no $m+1$ of them are collinear.

Lemma 2.8 ([7]). *For the (h,m)-arc $\mathcal{H}$, the following equations are hold:*

(i) $\sum_{i=0}^{m} \tau_i = q^2+q+1$;

(ii) $\sum_{i=1}^{m} i\tau_i = h(q+1)$;

(iii) $\sum_{i=2}^{m} \frac{i(i-1)}{2}\tau_i = \frac{h(h-1)}{2}$,

where τ_i represents the number of all i-secants of (h,m)-arc such that $\mathcal{H}\cap\tau = i$.

Definition 2.9 ([2]). A point P of PG$(2,q)$ is called a point of *index* 0 if it is not lying on the (h,m)-arc $\mathcal{H}$ and not on any m-secants of $\mathcal{H}$.

Theorem 2.10 ([4]). *For $2 = m = q+1$,*

(i) *the maximum size $z_m(2,q) \le (m-1)q+m$.*

(ii) *if $m \le q$ and equality took a place in (i), then m is a factor of q.*

Definition 2.11 ([2]). Suppose that π is a projective plane of order q. The sets of lines and points of π are denoted by R and p, respectively. Also, suppose that a function $f : P \to N$, where N is the set of the positive integers and zero, then $f(p)$ and the weight of $p\epsilon P$ are called the *non*-zero weighted points set of the plane. A function $F : R \longrightarrow Z^+$ can be defined by using the function f such that for any $r \in R$, $F(r) = \sum_{p\epsilon r} f(p).F(r)$ is called the weight of the line r.

Definition 2.12 ([2]). A $(k,n;f)$-arc of the plane π is a subset K of the points of the plane such that

(i) K is the support of f ;

(ii) $k = |K|$;

(iii) $n = \max\{F(r) : r \in R\}$.

Denote $\omega = \max_{p\epsilon P} f(p)$, V_i^j to the number of the lines that have weight of i through a point that has weight of$j$, and $W = \sum_{j=0}^{\omega} \mathcal{H}_j = \sum_{p\epsilon P} f(p)$. For a $(k,n;f)$-arc, we have the following important Lemma:

Lemma 2.13 ([3]). *For the weighted (k,n)-arcs in* PG$(2,q)$*, the following statements are holds:*

(i) $\omega = q$;

(ii) *If p is any point of the plane, then $\sum_{r\in[p]} F(r) = W + qf(p)$, where $[p]$ denote the set of lines through p ;*

(iii) *The weight W of a weighted (k,n)–arc satisfies $(n-q)(q+1) \le W \le (n-\omega)q+n$;*

(iv) *Let K be a weighted (k,n)-arc of type $(n-q,n)$, $n-q>0$ and let p be a point that has* *weight of s, then V_m^s and V_n^s can determine p and can be given as:*

$$V_{n-q}^s = \frac{q(n-s)-W+n}{q}$$

and

$$V_n^s = \frac{q(s-n+q)+W-n+q}{q};$$

(v) $q \equiv 0 \bmod(q)$;

(vi) $k = \sum_{j=1}^{2} l_j$;

(vii) *The characters of a weighted (k,n)-arcs K of type $(n-q,n)$ are given by*

$$t_{n-q} = \left[\frac{q+1}{q}\right]\left[\frac{n(q^2+q+1)}{q+1} - W\right]$$

and

$$t_n = \left[\frac{q+1}{q}\right]\left[W - \frac{(n-q)(q^2+q+1)}{q+1}\right].$$

Corollary 2.14 ([3]). *If $W = (n-q)(q+1)$, then a weighted (k,n)-arc is minimal and if $W = (n-\omega)+n$, then a weighted (k,n)-arc is maximal.*

Definition 2.15 ([9]). A $(k,n;f)$-arc is a monoidal when $\mathrm{Im} f = \{0,1,m\}$ and $l_m = 1$, with $m \geq 2$.

Principle of Duality 2.16 ([7]). For any space $S = \mathrm{PG}(n,q)$, there is a dual space S^*, whose points and primes are respectively primes and points of S. For any theorem true in S, there is an equivalent theorem true in S^*.

Lemma 2.17. *The existence of a $(k,n;f)$-arcs of type $(n-q,n)$, in* PG$(2,q)$ *with $q+1 < n < 2q+2$ requires $q \equiv 0 \bmod (q)$.*

Proof. Directly, from Lemma 2.13(v). □

Lemma 2.18 ([2]). *The existence of a $(k,n;f)$-arcs of type $(n-q,n)$, in* PG$(2,q)$ *with $q+1 < n < 2q+2$ requires $l_i = 0$, $i = 3$.*

We used Lemma 2.13(iii) *to get*

$$(n-q)(q+1) \leq W \leq (n-q)(q+1)+q$$

Lemma 2.19. *For a $(k,n;f)$-arcs of type $(n-q,n)$, in* PG$(2,q)$ *with W minimal ($W = (q+1)(n-q)$), we have*

$$V_{n-q}^0 = \frac{q(q+1)}{q}, \quad V_{n-q}^1 = \frac{q^2}{q}, \quad V_{n-q}^2 = \frac{q(q-1)}{q},$$

$$V_n^0 = 0, \quad V_n^1 = \frac{q}{q}, \quad V_n^2 = \frac{2q}{q}.$$

Proof. From Lemma 2.13(iv). □

Corollary 2.20. *There is no point of weight* 0 *on n-weighting lines of $(k,n;f)$-arcs of type $(n-q,n)$.*

For the case $l_0>0$, $l_1>0$, $l_2>0$, $l_i=0$, where $3\le i\le q$, we have the weight of the points of the $(k,n;f)$-arc is $\omega=2$, and by using the minimal case ($W=(n-q)(q+1)$) and by counting the number of lines of PG$(2,q)$, we find the following:

$$t_n+t_{n-q}=q^2+q+1.$$

By counting the number of n-weighting lines (t_n) and $(n-q)$-weighting lines (t_{n-q}), and counting the total incidence, it follows that

$$nt_n+(n-q)t_{n-q}=W(q+1)=(n-q)(q+1)^2.$$

Consequently, we get

$$t_n=(n-q), \tag{2.1}$$

$$t_{n-q}=(q^2+2q-n+1). \tag{2.2}$$

Lemma 2.21. *The n-weighting lines of $(k,n;f)$-arcs of type $(n-q,n)$ form a dual of t_n-arc in* PG$(2,q)$.

Proof. From Lemma 2.19, we have $V_n^2=2$, this means that there are no three n-weighting lines are concurrent. Then the number of n-weighting lines t_n form a dual of t_n-arc. □

On n-weighting lines, assume that there are α points and β points of weight one and weight two respectively. Then be calculation all the points in the n-weighting lines, it follows that:

$$\alpha+\beta=q+1$$

and calculation the weight of points on the n-weighting lines, we have

$$\alpha+2\beta=n.$$

Solving these two equations, we obtain

$$\alpha=2(q+1), \tag{2.3}$$

$$\beta=n-(q+1), \tag{2.4}$$

counting the incidences between the points of weight two and n-weighting lines, we get

$$l_2V_n^2=t_n\beta.$$

Making use of Lemma 2.19, equation (2.1) and equation (2.4) we obtain

$$l_2=\frac{(n-q)(n-q-1)}{2}. \tag{2.5}$$

Similarly, calculating the incidences between the points that have weight one and n-weighting lines, we have

$$l_1V_n^1=t_n\alpha.$$

Hence, by using Lemma 2.19, equation (2.2) and equation (2.3), we get

$$l_1=(n-q)(2q+2-n). \tag{2.6}$$

From equations (2.5) and (2.6), calculating the points in the plane, we have

$$l_0 + l_1 + l_2 = q^2 + q + 1, \tag{2.7}$$

$$l_0 = q^2 + q + 1 - (n-q)(2q+2-n) - \frac{(n-q)(n-q-1)}{2}. \tag{2.8}$$

Hence

$$l_0 = \frac{5q^2 + (5-4n)q + n^2 - 3n + 2}{2}. \tag{2.9}$$

Suppose that l be $(n-q)$-weighting lines and suppose that these lines have μ points, δ points, and γ points on it of weight 2, weight 1, and weight 0, respectively. Then, counting points on l gives

$$\mu + \delta + \gamma = q + 1 \tag{2.10}$$

and calculating the summation of the weights of points on l gives

$$2\mu + \delta = n - q, \tag{2.11}$$

where $n = 2q - u$, $u = -1, 0, 1, 2, \dots, q-2$. Hence the maximum values of μ and γ are $\frac{q-u}{2}$ and $\frac{q+u+2}{2}$, respectively.

3. Weighted (k,n)-arcs of Type $(n-q,n)$ and Maximum Size of (h,m)-arcs in PG$(2,q)$

Lemma 3.1. *There exists a maximum size $\left(\frac{q(q-1)}{2} + 1, \frac{q+1}{2}\right)$-arc in projective plane of order q.*

Proof. Put $n = 2q$, from equation (2.9) we get $l_0 = \frac{q(q-1)}{2}$.
Let l be a line of weighting $(n-q)$. Suppose that there are μ points of weight two, δ points of weight one and γ points of weight zero, we have

$$\mu + \delta + \gamma = q + 1, \tag{3.1}$$

$$2\mu + \delta = q. \tag{3.2}$$

The only non-negative integers solutions are given in Table 1.

Table 1

μ	δ	γ
$\frac{q-1}{2}$	1	$\frac{q+1}{2}$
$\frac{q-3}{2}$	3	$\frac{q-1}{2}$
$\vdots$	$\vdots$	$\vdots$
0	q	1

From the solutions above we get that the points of weight zero form a $\left(\frac{q(q-1)}{2} + 1, \frac{q+1}{2}\right)$-arc of type $\left(\tau_{\frac{q+1}{2}} = \frac{q(q+1)}{2}, \tau_{\frac{q-1}{2}} = \frac{q(q-1)}{2}, \tau_1 = 2q - n + 1, \tau_0 = n - q\right)$. □

Lemma 3.2. *There exist a maximum size $((q-1)^2, q-1)$-arc in projective plane of order q.*

Proof. Put $n = q+3$, from equation (2.9), we get $l_0 = (q-1)^2$.
Let l be a line of weighting $(n-q)$. Suppose that there are μ points of weight two, δ points of weight one and γ points of weight zero, we have

$$\mu+\delta+\gamma = q+1,$$
$$2\mu+\delta = 3.$$

The only non-negative integers solutions are given in Table 2.

Table 2

μ	δ	γ
1	1	$q-1$
0	3	$q-2$

From the solutions above we get that the points of weight zero form a $((q-1)^2, q-1)$-arc of type $(\tau_{q-1} = 3(q-1)$, $\tau_{q-2} = (q-1)^2$, $\tau_0 = n-q)$.

Lemma 3.3. *There exist a maximum size $(q(q-1),q)$-arc in projective plane of order q.*

Proof. Put $n = q+2$, from equation (2.9), we get $l_0 = q(q-1)$.

Let l be a line of weighting $(n-q)$. Suppose that there are μ points of weight two, δ points of weight one and γ points of weight zero, we have

$$\mu+\delta+\gamma = q+1,$$
$$2\mu+\delta = 2.$$

The only non-negative integers solutions are given in Table 3.

Table 3

μ	δ	γ
1	0	q
0	2	$q-1$

From the solutions above we get that the points of weight zero form a $(q(q-1),q)$-arc of type $(\tau_q = q-1, \tau_{q-1} = q^2, \tau_0 = n-q)$.

Since $k = \sum_{j=1}^{2} l_j$ and $n = 2q-u$, where $u = -1,0,1,\dots,q-2$.

Hence we deduce the following theorem. □

Theorem 3.4. *There exist a $\left(\frac{(q-u)(q+u+3)}{2}, 2q-u; f\right)$-arc of type $(q-u, 2q-u)$ in* PG$(2,q)$ *with the* $Im\,f = \{0,1,2\}$ *and the points of weight zero are* $\frac{q(q-1)}{2}+1$, $(q-1)^2$ *and* $q(q-1)$.

4. Conclusion

In this paper, we showed that the order of weighted (k,n)-arcs can be generalized into any order of a prime number, and this study has not been done before. In fact, all the previous studies that mentioned in our paper were about specific orders such as PG$(2,3)$, PG$(2,7)$, PG$(2,9)$ and so on. In addition, we were able to find a maximal (h,m)-arcs in PG$(2,q)$. Finally, we proved that a $\left(\frac{q(q-1)}{2}+1,\frac{q+1}{2}\right)$-arc is a maximal arc in PG$(2,q)$.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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