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On τ_M-Semilocal Modules and Rings

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Abstract. Let τ_M be any preradical for $\sigma[M]$ and N any module in $\sigma[M]$. In [2], Al-Takhman, Lomp and Wisbauer defined and studied the concept of τ_M-supplemented module. In this paper we define the concept of weakly τ_M-supplemented module and investigate some properties of such modules. We show that weakly τ_M-supplemented module N is τ_M-semilocal (i.e., $N/\tau_M(N)$ is semisimple) and that R is a τ-semilocal ring if and only if $_RR$ (or R_R) is weakly τ_M-supplemented.

1. Introduction

Throughout this paper R will denote an arbitrary associative ring with identity and all modules will be unitary right R-modules. Let $M \in$ Mod-R. By $\sigma[M]$ we mean the full subcategory of Mod-R whose objects are submodules of M-generated modules. For any module M, τ_M will denote a preradical in $\sigma[M]$. Let $N \in \sigma[M]$ be a module. Following [2], a submodule $K \subseteq N$ is called τ_M-supplement provided there exists some $U \subseteq N$ such that $U + K = N$ and $U \cap K \subseteq \tau_M(K)$. N is called τ_M-*supplemented* if each of its submodule has a τ_M-supplement in N. N is called *amply* τ_M-*supplemented*, if for all submodules K and L of N with $K + L = N$, K contains a τ_M-supplement of L in N.

A module $N = \sum_{i\in I} N_I$ is called irredundant sum if for $j \in I$ we have $N \neq \sum_{i\neq j} N_i$. A preradical τ for $\sigma[M]$ is said to be idempotent, if for each $N \in \sigma[M]$, $\tau(\tau(N)) = \tau(N)$. $K \ll N$ means that K is small in N (i.e. $\forall\ L \lneq N$, $L + K \neq N$). By $K \leq_e N$ we mean that K is an essential submodule of N (i.e. $\forall\ 0 \neq L \leq N$, $L \cap K \neq 0$).

In Section 2, we study some properties of τ_M-supplemented and amply τ_M-supplemented modules. In [6], Wang and Ding defined the concept of weakly generalized supplemented modules. In Section 3, we define weakly τ_M-supplemented modules, it is clear that every weakly τ_M-supplemented

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is weakly generalized supplemented module. In Section 4, we show that every weakly τ_M-supplemented module N is τ_M-semilocal (i.e., $N/\tau_M(N)$ is semisimple) and that R is a τ-semilocal ring if and only if ${}_RR$ (or R_R) is weakly τ_M-supplemented.

Lemma 1.1 (See [3, Proposition 5.20]). *Suppose that $K_1 \leq N_1 \leq N$, $K_2 \leq N_2 \leq N$, and $N = N_1 \oplus N_2$. Then $K_1 \oplus K_2 \leq_e N_1 \oplus N_2$ if and only if $K_1 \leq_e N_1$ and $K_2 \leq_e N_2$.*

2. τ_M-Supplemented and τ_M-Amply Supplemented Modules

In [2], Al-Takhman, Lomp and Wisbauer defined and studied the concept of τ_M-supplemented module. In this section we investigate some properties of such modules.

Proposition 2.1. *Let $N \in \sigma[M]$ be a τ_M-supplemented module and L a submodule of N with $L \cap \tau_M(N) = 0$. Then L is semisimple. In particular, a τ_M-supplemented module N with $\tau_M(N) = 0$ is semisimple.*

Proof. Let L' be any submodule of L. Since N is a τ_M-supplemented module, there exists $L'' \leq N$ such that $L' + L'' = N$ and $L' \cap L'' \subseteq \tau_M(L'')$. Thus $L = L \cap N = L \cap (L' + L'') = L' + L \cap L''$. Since $L' \cap L'' \subseteq \tau_M(L'')$ and $L' \cap L \cap L'' = L' \cap L'' \subseteq L \cap \tau_M(L'') \subseteq L \cap \tau_M(N) = 0$, $L = L' \oplus (L \cap L'')$. So L is semisimple. □

Proposition 2.2. *Let $N \in \sigma[M]$ be a τ_M-supplemented module. Then $N = H \oplus L$, where H is semisimple and L is a module with essential preradical.*

Proof. For $\tau_M(N)$, there exists $H \leq N$ such that $H \cap \tau_M(N) = 0$ and $H \oplus \tau_M(N) \leq_e N$. Since N is a τ_M-supplemented module, there exists $L \leq N$ such that $H + L = N$ and $H \cap L \subseteq \tau_M(L)$. Since $H \cap L = H \cap (H \cap L) \subseteq H \cap \tau_M(L) \subseteq H \cap \tau_M(N) = 0$, $N = H \oplus L$. By Proposition 2.1, H is semisimple. Thus $\tau_M(N) = \tau_M(H) \oplus \tau_M(L) = \tau_M(L)$. Since $H \oplus \tau_M(N) \leq_e N = H \oplus L$, i.e., $H \oplus \tau_M(L) \leq_e N = H \oplus L$, $\tau_M(L) \leq_e L$ (Lemma 1.1). This completes the proof. □

Let N be a module and $K \leq N$. K is said to has ample τ_M-supplement in N if for every submodule L such that $N = K + L$, K has a τ_M-supplement in N.

Proposition 2.3. *Let $N \in \sigma[M]$ be a module and $N = U_1 + U_2$. If U_1, U_2 have ample τ_M-supplements in N, then also $U_1 \cap U_2$ has.*

Proof. Let $V \leq N$ and $U_1 \cap U_2 + V = N$. Then $U_1 = U_1 \cap U_2 + (V \cap U_1)$ and $U_2 = U_1 \cap U_2 + (V \cap U_2)$, so $N = U_1 + V \cap U_2$ and $N = U_2 + V \cap U_1$. Since U_1, U_2 have amply τ_M-supplement module in N, there exist $V_2' \leq V \cap U_2$ and $V_1' \leq V \cap U_1$ such that $U_1 + V_2' = N$ and $U_1 \cap V_2' \subseteq \tau_M(V_2')$, and $U_2 + V_1' = N$ also $U_2 \cap V_1' \subseteq \tau_M(V_1')$. Thus $V_1' + V_2' \leq V$ and $U_1 = U_1 \cap U_2 + V_1'$ and $U_2 = U_1 \cap U_2 + V_2'$. Therefore $(U_1 \cap U_2) + (V_1' + V_2') = N$ and $(U_1 \cap U_2) \cap (V_1' + V_2') = (U_2 \cap V_1') + (U_1 \cap V_2') \subseteq \tau_M(V_1' + V_2')$. This completes the proof. □

Lemma 2.4. *Let U, V be submodules of $N \in \sigma[M]$ and V a τ_M-supplement submodule of U in N. If U is a maximal submodule of N, then $U \cap V = \tau_M(V)$ is a unique maximal submodule of V.*

Proof. Since $V/(U \cap V) \simeq N/U$, $U \cap V$ is a maximal submodule of V, and hence $\tau_M(V) \subseteq U \cap V$. Since $U \cap V \subseteq \tau_M(V)$, $U \cap V = \tau_M(V)$, as desired. □

Definition 2.5. A module H is called τ_M-hollow if for every proper submodule H' of H, $H' \subseteq \tau_M(H)$.

Theorem 2.6. *Let $N \in \sigma[M]$ be a module. N is a sum of τ_M-hollow submodules and $\tau_M(N) \ll N$ if and only if N is an irredundant sum of local modules and $\tau_M(N) \ll N$.*

Proof. Let $N = \sum_I L_i$ where $L_i (i \in I)$ is τ_M-hollow submodule of N. Then $N/\tau_M(N) = \sum_I (L_i + \tau_M(N))/\tau_M(N)$. Since $\tau_M(L_i) \subseteq L_i \cap \tau_M(N)$ and $(L_i + \tau_M(N))/\tau_M(N) \simeq L_i/(L_i \cap \tau_M(N))$, these factors are either simple or zero. Thus we have $N/\tau_M(N) = \bigoplus_{I'} (L_i + \tau_M(N))/\tau_M(N)$. Therefore $N = \sum_{I'} L_i$ is an irredundant sum of local submodules $L_i (i \in I' \subseteq I)$ (for $\tau_M(N) \ll N$), as required. □

3. τ_M-Weakly Supplemented Modules

Definition 3.1. A submodule $K \subseteq N$ is called weak τ_M-supplement provided there exists $U \subseteq N$ such that $U + K = N$ and $U \cap K \subseteq \tau_M(N)$.

Definition 3.2. A module $N \in \sigma[M]$ is said to be a weakly τ_M-supplemented module if for any submodule $N' \leq N$,there exists $L \leq N$ such that $N = N' + L$ and $N' \cap L \subseteq \tau_M(N)$.

Proposition 3.3. *Let $N \in \sigma[M]$ be a weakly τ_M-supplemented module. Then:*

(i) *If τ_M is idempotent, then every τ_M-supplement submodule of N is a weakly τ_M-supplemented module.*

(ii) *Every factor module of N is a weakly τ_M-supplemented module.*

Proof. (i) Let K be a τ_M-supplement submodule in N. For any submodule $N' \leq K$, since N is a weakly τ_M-supplemented module, there exists $L \leq N$ such that $N = N' + L$ and $N' \cap L \subseteq \tau_M(N)$. Thus $K = K \cap N = K \cap (N' + L) = N' + (K \cap L)$ and $N' \cap (K \cap L) = N' \cap L = K \cap (N' \cap L) \subseteq K \cap \tau_M(N) = \tau_M(K)$. Therefore K is a weakly τ_M-supplemented module.

(ii) Let N' be any submodule of N and L/N' any submodule of N/N'. For $L \leq N$, there exists $K \leq N$ such that $L + K = N$ and $K \cap L \subseteq \tau_M(N)$ since N is a weakly τ_M-supplemented module. Thus $N/N' = L/N' + (K + N')/N'$. Let $f : N \to N/N'$ be a canonical epimorphism. Since $K \cap L \subseteq \tau_M(N)$, $(L/N') \cap ((K + N')/N') = (L \cap (K + N'))/N' = (N' + (K \cap L))/N' = f(L \cap K) \subseteq f(\tau_M(N)) \subseteq \tau_M(N/N')$. This completes the proof. □

Lemma 3.4. *Let $K, N_1 \leq N \in \sigma[M]$ and N_1 a weakly τ_M-supplemented module. If $N_1 + K$ has a weak τ_M-supplement in N, then also K has.*

Proof. By assumption, there exists $N' \leq N$ such that $(N_1 + K) + N' = N$ and $N' \cap (N_1 + K) \subseteq \tau_M(N)$. Since N_1 is a weakly τ_M-supplemented module, there exists a submodule $L \leq N_1$ such that $N_1 \cap (N' + K) + L = N_1$ and $L \cap (N' + K) \subseteq \tau_M(N_1)$. Thus $N = K + N' + L$ and $K \cap (N' + L) \leq (K + N_1) \cap N' + L \cap (N' + K) \subseteq \tau_M(N)$, that is, $N' + L$ is a weak τ_M-supplement of K in N. □

Proposition 3.5. *Let $N = N_1 + N_2$. If N_1 and N_2 are weakly τ_M-supplemented module, then N is a weakly τ_M-supplemented module.*

Proof. Let N' be a submodule of N. Since $N_1 + N_2 + N' = N$ trivially has a weak τ_M-supplement in N, $N_2 + N'$ has a weak τ_M-supplement in N (Lemma 3.4). Thus N' has a weak τ_M-supplement in N (Lemma 3.4). So N is a weakly τ_M-supplemented module. □

Corollary 3.6. *Every finite sum of weakly τ_M-supplemented modules is weakly τ_M-supplemented.*

A module N is called τ_M-*semilocal* if $N/\tau_M(N)$ is semisimple.

Theorem 3.7. *Let $N \in \sigma[M]$ be a module and $\tau_M(N) \ll N$. Then the following statements are equivalent.*

(i) *N is a weakly τ_M-supplemented module.*
(ii) *N is τ_M-semilocal.*
(iii) *There is a decomposition $N = N_1 \oplus N_2$ such that N_1 is semisimple, $\tau_M(N) \leq_e N_2$ and N_2 is τ_M-semilocal.*

Proof. (i)$\Longrightarrow$(ii) Let L be any submodule of N containing $\tau_M(N)$. Since N is weakly τ_M-supplemented module, there exists $N' \leq N$ such that $N' + L = N$ and $N' \cap L \leq \tau_M(N)$. Thus $N/\tau_M(N) = L/\tau_M(N) + (N' + \tau_M(N))/\tau_M(N)$ and $L/\tau_M(N) \cap (N' + \tau_M(N))/\tau_M(N) = (L \cap N' + \tau_M(N))/\tau_M(N) = 0$. So $N/\tau_M(N) = L/\tau_M(N) \oplus (N' + \tau_M(N))/\tau_M(N)$, as required.

(ii)$\Longrightarrow$(i) For any submodule $N' \leq N$, since $N/\tau_M(N)$ is semisimple, there exists a submodule $L \leq N$ containing $\tau_M(N)$ such that $N/\tau_M(N) = (N' + \tau_M(N))/\tau_M(N) \oplus L/\tau_M(N)$. Thus $N = N' + \tau_M(N) + L$. Since $\tau_M(N) \ll N$, $N = N' + L$. $N' \cap L \subseteq \tau_M(N)$ is obvious.

(ii)$\Longrightarrow$(iii) Let N_1 be a complement of $\tau_M(N)$ in N. Then $N_1 \simeq (N_1 \oplus \tau_M(N))/\tau_M(N)$ is a direct summand of $N/\tau_M(N)$, and hence it is semisimple. Therefore, there exists a semisimple submodule $N_2/\tau_M(N)$ such that $(N_1 \oplus \tau_M(N))/\tau_M(N) \oplus N_2/\tau_M(N) = N/\tau_M(N)$. Thus $N_1 + N_2 = N$ and $N_1 \cap N_2 \leq \tau_M(N) \cap N_1 = 0$ implies $N = N_1 \oplus N_2$. Since $N_1 \oplus \tau_M(N) \leq_e N = N_1 \oplus N_2$, $\tau_M(N) \leq_e N_2$ by Lemma 1.1.

(iii)$\Longrightarrow$(ii) It is clear. $\square$

Corollary 3.8. *Let $N \in \sigma[M]$ be a module with $\tau_M(N) = 0$. Then N is weakly τ_M-supplemented if and only if N is semisimple.*

Proof. This follows by Theorem 3.7. $\square$

4. τ_M-Semilocal Modules and Rings

Let $Gen(M)$ denote the class of M-generated modules.

Theorem 4.1. *The following statements for a finitely generated module M are equivalent:*

(a) *M is τ_M-semilocal;*
(b) *Any $N \in Gen(M)$ is τ_M-semilocal;*
(c) *Any $N \in Gen(M)$ is a direct sum of a semisimple module and a τ_M-semilocal module with essential radical;*
(d) *Any $N \in Gen(M)$ with small preradical is weakly τ_M-supplemented.*

Proof. (a)$\Longrightarrow$(b) For every $N \in Gen(M)$ there exists a set Λ and an epimorphism $f : M^{(\Lambda)} \to N$. Since $f(\tau_M(M^{(\Lambda)})) \subseteq \tau_M(N)$ and $M^{(\Lambda)}/\tau_M(M^{(\Lambda)}) \simeq (M/\tau_M(M))^{(\Lambda)}$ always holds we get an epimorphism $\overline{f} : (M/\tau_M(M))^{(\Lambda)} \to N/\tau_M(N)$. Hence N is τ_M-semilocal.

(b)$\Longrightarrow$(a) It is trivial.

(d)$\Longleftrightarrow$(b)$\Longleftrightarrow$(c) By Theorem 3.7. $\square$

The ring R is called *τ_M-semilocal* if ${}_RR$ (or R_R) is a τ_M-semilocal R-module.

Proposition 4.2. *For a ring R the following statements are equivalent:*

(a) *${}_RR$ is weakly τ_M-supplemented;*
(b) *R is τ_M-semilocal;*
(c) *R_R is weakly τ_M-supplemented.*

Proof. Apply Theorem 3.7 and use that ' τ_M-semilocal' is a left-right symmetric property. $\square$

Theorem 4.3. *For any ring R the following statements are equivalent:*

(a) *R is τ_M-semilocal;*
(b) *Every left R-module is τ_M-semilocal;*
(c) *Every left R-module is the direct sum of a semisimple module and a τ_M-semilocal module with essential preradical;*
(d) *Every left R-module with small preradical is weakly τ_M-supplemented.*

Proof. It follows from Theorem 4.1. $\square$

We call K *τ_M-cover* of a module N if there exists an epimorphism $f : K \to N$ such that $Ker(f) \subseteq \tau_M(K)$. K is called a *τ_M-projective cover, τ_M-free cover* of N respectively if K is a τ_M-cover of N and K is a projective, free module resp.

Proposition 4.4. *Every finitely generated R-module over a τ_M-semilocal ring is a direct summand of a module having a finitely generated τ_M-free cover.*

Proof. Let N be a finitely generated R-module. Then there exists a number k and a epimorphism $f : R^k \to N$. Since R is τ_M-semilocal, R^k is weakly τ_M-supplemented. Hence $Ker(f)$ has a weak τ_M-supplement $L \subseteq R^k$. Thus the natural projection $R^k \to N \oplus (R^k/L)$ with kernel $Ker(f) \cap L \subseteq \tau_M(R^k)$ implies that R^k is a τ_M-projective cover for $N \oplus (R^k/L)$. □

Lemma 4.5. *Let R be a ring, $r, a \in R$ and $b := 1 - ra$. Then $Ra \cap Rb = Rab$.*

Proof. See [4, Lemma 3.4]. □

Proposition 4.6. *For any ring R the following statements are equivalent:*

(a) *Every principal left ideal of R has a weak τ_M-supplement in ${}_RR$;*
(b) *$R/\tau_M(R)$ is von Neumann regular;*
(c) *Every principal right ideal of R has a weak τ_M-supplement in R_R.*

Proof. (a)$\Longrightarrow$(b) Let $a \in R$. By assumption there exists a weak τ_M-supplement $I \subset R$ of Ra. Then there exist $b \in R$ and $x \in I$ such that $x = 1 - ba$. Moreover, by Lemma 4.5, $Rax = Ra \cap Rx \subseteq Ra \cap I \subseteq \tau_M(R)$ implies $ax = a - aba \in \tau_M(R)$. Thus $R/\tau_M(R)$ is von Neumann regular.

(b)$\Longrightarrow$(a) For any $a \in R \setminus \tau_M(R)$ we get an element $b \in R \setminus \tau_M(R)$ such that $a - aba \in \tau_M(R)$. Then $Ra \cap R(1 - ba) = R(a - aba) \subseteq R\tau_M(R) \subseteq \tau_M(R)$ (Lemma 4.5). Hence $R(1 - ba)$ is a weak τ_M-supplement of Ra in ${}_RR$.

(b)$\Longrightarrow$(c) Analogous. □

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