Communications in Mathematics and Applications
Vol. 10, No. 3, pp. 519–530, 2019
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v10i3.1203

Research Article

Semi Unit Graphs of Commutative Semi Rings

Yaqoub Ahmed* and M. Aslam

Department of Mathematics, GC University, Lahore, Pakistan
***Corresponding author:** yaqoubahmedkhan@gmail.com

Abstract. In this article, we introduce semi unit graph of semiring S denoted by $\xi(S)$. The set of all elements of S are vertices of this graph where distinct vertices x and y are adjacent if and only if $x+y$ is a semiunit of S. We investigate some of the properties and characterization results on connectedness, distance, diameter, girth, completeness and connectivity of $\xi(S)$.

Keywords. Semirings; Semiunits; k-ideals; Graphs

MSC. 16Y60; 05C25

Received: February 26, 2019 **Revised:** May 1, 2019 **Accepted:** Accepted 13, 2019

1. Introduction

The concepts of the graph to the elements (zero-divisors) of a ring was introduced by Beck in [8] and discussed the coloring of a commutative ring. The zero-divisor graph of a commutative ring has been studied extensively by several authors like Anderson [2], and Atani [3,5]. Moreover, Atani and others establish the properties of zero divisor graphs in semirings and introduced unit graph in semirings in [3,5]. In this paper, our focus is on semi unit elements of semirings and their graphs.

For the sake of completeness, we state some definitions and notations used throughout this paper. In this paper, we refer S to be commutative semi ring with multiplicative identity and absorbing zero, unless mentioned otherwise. A non-zero element $a \in S$ is said to be semi unit if there exists, $r,s \in S$ such that $1+r.a = s.a$ (c.f. [7]). The set of all semi units of S is denoted by S_u and the set of all non-semi units is denoted by N_u in this paper. Every unit is a semi unit, by taking $r = 0$ in definition. In a ring every semi unit is a unit. An ideal K is said to be k-ideal (Subtractive ideal) such that if $x, x+y \in K$ then $y \in K$ (c.f. [10]). A k-ideal which is also maximal ideal is called maximal k-ideal. P is maximal k-ideal of S if and only if S/P is a semi field [7]. Let P be an ideal of a semi ring S, P is a prime k-ideal of S if and only if S/P is a semi domain [4]. Let S be a semi ring with non-zero identity. S is said to be a local semi ring if and only if S has a unique maximal k-ideal. The k-closure $cl(I)$ of ideal I is defined by $cl(I) = \{a \in S; a + c = d$ for some $c, d \in I\}$ which is indeed the smallest k-ideal of S containing I [10].

An ideal I of semiring S is called a partitioning ideal(Q-ideal) if there exists a subset Q of S such that: $S = \cup\{q + I : q \in Q\}$. If $q_1, q_2 \in Q$ then $(q_1 + I) \cap (q_2 + I) \neq$ if and only if $q_1 = q_2$. If I is a Q-ideal of a semiring S then I is a k-ideal of S, by [12, Lemma 2]. For basic concepts of semirings, we refer [1, 9, 11].

There are some special families of graphs as complete graph is a simple graph in which any two vertices are adjacent. A graph is connected if there is a path between any two vertices. A graph is bipartite if its vertex set can be partitioned into two subsets X and Y so that every edge has one end in X and one end in Y. If ξ is simple and every vertex in X is joined to every vertex in Y, then ξ is called a complete bipartite graph. An isomorphism between two simple graphs X and Y as a bijection $\theta : V(X) \to V(Y)$ which preserves adjacency. For undefined terms related to graph theory, we refer [16].

2. Results of Semi Units

Firstly, we discuss some results on semiunit elements, which are helpful to study the graphs of semiunits.

Lemma 2.1 ([7]). *Let S be a semiring and let $a \in S$. Then a is a semi-unit of S if and only if a lies outside each k-maximal ideal of S.*

Lemma 2.2 ([7]). *If $x \in S$ be a commutative semi ring, then $cl(Sx)$ is a k-ideal of S.*

Proposition 2.3. *Every ideal generated by a non-semi unit in a commutative semiring S is proper ideal and contain in some k-maximal ideal.*

Proof. Consider a be a non-semi unit in S. It is observed that Sa is an ideal. Also, $Sa \subseteq cl(Sa)$ which is k-ideal, whereas every k-ideal of semi ring contained in some k-maximal ideal by ([15, Corollary 2.2]), so there must exist some k-maximal ideal M (say) such that $\langle a \rangle = Sa \subseteq cl(Sa) \subseteq M$ implies that $\langle a \rangle = Sa \subseteq M$. □

Proposition 2.4. *The set of semi units S_u of semi ring S are closed under multiplication.*

Proof. Consider $a, b \in S_u$, such that

$$1 + r.a = s.a \quad \text{and} \quad 1 + t.b = u.b \quad \text{for some } r, s, t, u \in S.$$

Let $a.b$ is a non-semiunt. By using Proposition 2.1, we get that there exist a proper ideal $\langle ab\rangle$, which contain in some k-maximal ideal M such that $\langle ab\rangle \subseteq M$ implies that $ab \in M$. M is an ideal therefore $sab, rab \in M$, for all $r,s \in S$. Also, we have $1+r.a = s.a$ or $b+r.a.b = s.a.b \in M$ this implies that $b+r.a.b \in M$, since M is k-ideal therefore $b \in M$ which is a contradiction. □

Theorem 2.5. *Let the set of all non-semiunits N_u is not an ideal then N_u is not closed under addition.*

Proof. By hypothesis, we have N_u is not an ideal.
Let $a \in S$ and $x \in N_u$. We will show that $a.x \in N_u$. Suppose on contrary that $a.x \in S_u$. It gives

$$1+(a.x).r = (a.x).s \quad \text{for some } r,s \in S$$

that is $1+x.(a.r) = x.(a.s)$ or $1+x.r_1 = x.s_1$, where $r_1 = a.r$, $s_1 = a.s \in S$, this implies that x is semi unit, which is contradiction. Therefore, $a.x \in N_u$. Then by hypothesis, we have some distinct non-semi units $x, y \in N_u$ such that $x+y \in S_u$. □

3. Semi Unit Graphs

Consider S is commutative semi ring and S_u denotes of set of all semi unit elements in S. We develop a semi unit graph, denoted by ξ, by taking every element as vertex. If $x, y \in S$ $(x \neq y)$ then x and y are adjacent if $x+y \in S_u$, otherwise they are disjoint vertices therefore we are considering simple graphs. We shall discuss about its properties, characteristics and different shapes of Semi Unit graphs in this section.

Theorem 3.1. *Consider a Semi ring S, then the Semi unit graph is complete if and only if $S_u = S - \{0\}$.*

Proof. If $S_u = S - \{0\}$, then S_u is zero-sum free, since S is semi ring otherwise it will be a ring (c.f. [6, Lemma 2.1]). This shows that S_u is closed under addition, so every two vertices of S_u are adjacent. Also 0 is adjacent with all other elements of S as $0+x = x \in S_u$, therefore graph is complete.

Conversely, suppose ξ is complete graph then every vertex $x \in S$ of graph is connected with all other vertices. As $0 \in S$, therefore 0 is also adjacent to every $x \in S$ and $0+x = x$ then by definition of semiunit graph $x \in S_u$, this implies that all non-zero elements $x \in S_u$, that is $S_u = S - \{0\}$. □

Theorem 3.2. *Let S be a Semi ring with multiplicative identity 1. If N_u makes k-ideal then ξ is connected graph.*

Proof. Consider S is semi ring with and the set of all non-semi units N_u makes k-ideal. Let $x, y \in N_u$, then $x+y \in N_u$, that there is no edge between them. Now consider $x \in N_u$, and $u \in S_u$ then $x+u$ must be semi unit. Suppose on contrary that $x \in N_u$, and $x+u \in N_u$ then by hypothesis $u \in N_u$ which is contradiction, so $x+u \in S_u$, there is always an edge between a semi unit and non- semi unit in this semi ring, so all vertices must be connected at least through identity 1. Hence graph is connected. □

Proposition 3.3 ([7]). *Let S be a semiring. Then S is a local semiring if and only if the set of non-semi-unit elements of S is a k-ideal.*

From previous Theorem 3.2 and Proposition 3.3, we can easily conclude the following result.

Corollary 3.4. *The Semi unit graph of Local Semi ring S ($|S| \geq 2$), with identity 1 is always connected.*

Theorem 3.5. *If S be a commutative semi ring without identity, the semi unit graph is totally disconnected.*

Proof. Let S be a commutative semi ring without identity then $S_u = \emptyset$ and suppose on contrary that there exist elements $a, b \in S$, such that there is an edge between a and b, that $a + b$ is semi unit, which is a contradiction. Hence graph is totally disconnected. □

Proposition 3.6. *Let S be a commutative semi ring then $1 \leq |\xi| < \infty$ (i.e. semiunit graph is finite), if and only if S is finite or not a semi field.*

Proof. For S is finite, then it is trivial.
If S is infinite, then $|\xi|$ is finite if number of semi units $|S_u|$ are finite, since addition in S is well defined. If $U(S)$ is set of all unit elements of S then $U(S) \subseteq S_u$ and here $|\xi|$ is finite which tells that S cannot be infinite semi field (otherwise every element is unit so also semi unit in semi field).
Conversely, Suppose that S is infinite semi field then $S_u = U(S) = \infty$ then $|\xi| = \infty$ since 0 is adjacent with all semiunits (units). Thus $1 \leq |\xi| < \infty$ if and only if S is not a semi field. □

Proposition 3.7. *If two semi ring R, S are isomorphic then their graphs $\xi(R)$ and $\xi(S)$ are also isomorphic.*

Proof. Clearly, $|R| = |S|$, therefore number of vertices are equal. Now, we will prove that the adjacency of vertices are also preserved. Firstly, we shall show that image of semi unit is also semi unit under the isomorphism between R and S. Consider that isomorphism between R and S, $f : R \to S$, such that $f(r) = s$. Let a is semi unit in R. Then, $1 + va = wa$, for $v, w \in R$, therefore

$$f(1 + va) = f(wa) \Longrightarrow f(1) + f(v)f(a) = f(w)f(a)$$

implies that $1_s + v_s a_s = w_s a_s$, where 1_s is identity in S, $v_s, w_s, a_s \in S$. This shows that a_s is semiunit in S and f maps semiunit of R to semiunit of S.

Now to check the edges, if $x, y \in R$ such that $x+y \in (S_u)_R$, semi units in R. Then, $f(x), f(y) \in S$, such that $f(x) + f(y) = f(x + y) \in (S_u)_S$, semi units in S. Hence, whenever R, S are isomorphic then so is there semi unit graphs. □

By Theorem 2.5, the following proposition is straight forward.

Proposition 3.8. *Let S be a semi ring with identity with N_u is non-ideal.Then there exist some non adjacent vertices in N_u*

Before establishing next results, we state some definitions of graph theory (see [16]). The **distance (or length)** between two vertices x, y in a graph is the number of edges in a shortest path connecting them and denoted by $d(x, y)$. The maximum distance in the graph G is called **diameter** of graph which is denoted by $diam(G)$. The **girth** of a graph is the length of a shortest cycle contained in the graph.

Proposition 3.9. *Let S be a semi ring and N_u is an ideal. If $a, b \in S$, $a \neq b$ then the length of the path between a to b is* $1, 2, 3, 4$, *otherwise a is not adjacent with b, that is* $d(a, b) \in \{1, 2, 3, 4$ *or* $\infty\}$.

Proof. We shall discuss three cases to prove the above theorem

Case 1: If $a, b \in S_u$ then $a + b \in S_u$ or $a + b \in S_u$. If $a + b \in S_u$ then $d(a, b) = 1$. If $a, b \in S_u$ such that $a + b \notin S_u$ then 0 is adjacent to both a and b, so there is path $a — 0 — b$ between a and b, therefore $d(a, b) = 2$.

Case 2: If $a \in S_u$, $b \notin S_u$. If $a + b \in S_u$ then $d(a, b) = 1$. If there exist some $c \in S_u$ such that $b + c \in S_u$ then there is path $b — c — 0 — a$ with $d(a, b) = 3$. On the other hand, if for all $c \in S_u$ such that $b + c \notin S_u$, then $d(a, b) = \infty$. Similarly, if $a \notin S_u$, $b \in S_u$, then the same situation arises.

Case 3: If $a, b \in N_u$ such that $a + b \in S_u$ then $d(a, b) = 1$.
If $a, b \in N_u$ such that $a + b \in N_u$, and there is some possibility to find $c, d \in S_u$ such that $a + c \in S_u$ and $b + d \in S_u$, then there is shortest path $a — c — 0 — b — d$ with $d(a, b) = 4$.
Otherwise, if there is no path to connect a and b then $d(a, b) = \infty$. □

Example 3.10. **(1)** In every semi field S, provided that it is not a field then every non-zero element is a unit hence semi unit in S, and S_u is closed under addition that is for all $a, b \in S_u = S - \{0\}$, there is $a + b \in S_u = S - \{0\}$ so $d(a, b) = 1$ for all $a, b \in S$, therefore the graph is complete.

(2) Consider a set $S = \{0, 1, 2, 3, 4, 5\}$ with binary operation + as maximum and × as minimum. Here 0 treats as zero of the set while 5 behaves as multiplicative identity. This, clearly makes a S to be semi ring with multiplicative identity. $S_u = \{5\}$ and $N_u = \{0, 1, 2, 3, 4\}$. Consider two vertices $1, 4$ there is no direct edge between them but if we take the paths $5 — 1$ and $5 — 4$ them there is a path $1 — 5 — 4$, hence $d(1, 4) = 2$.

(3) Consider $S = Z_6$ with usual addition and multiplication, where $S_u = \{1, 5\}$, if we consider the path between 0 and 3, then there is no edge in them but a path exists by taking $0 — 1 — 4 — 3$ or $0 — 5 — 2 — 3$, so $d(a, b) = 3$.

(4) Consider a semi ring $N_0 = \{0, 1, 2, 3, \ldots\}$. Here 1 is identity and only semi unit in this semi ring, $d(1, i) = \infty$ for all $i \in N$ and its graph is disconnected.

Proposition 3.11. *Let S be a finite local semi ring with identity, then its semi unit graph has length of path* $d(a, b) \in \{1, 2\}$ *for all* $a, b \in S$.

Proof. (1) If $a, b \in S_u$ such that $a + b \in S_u$, then $d(a, b) = 1$.

If $a,b \in S_u$ such that $a+b \in N_u$ then there is path a — 0 — b, so $d(a,b)=2$.

(2) If $a \in S_u$, $b \in N_u$. As S is local semiring therefore N_u is maximal k-ideal (c.f. Proposition 3.3) this implies that $a+b \in S_u$. Hence $d(a,b)=1$. Similarly, when $b \in S_u$, $a \in N_u$.

(3) If $a,b \in N_u$ then $a+b \in N_u$. Then for some $u \in S$ there exist a path a — u — b for some $u \in S_u$. Hence $d(a,b)=2$. □

Theorem 3.12. (a) *If S is local semi ring with zero and 0 is the only non-semi unit (or a semi field with 0) then $diam\xi = sup\{d(x,y), x,y \in S\} = 1$.*

(b) *If S is finite local semi ring and $|N_u| \geq 2$, then $diam\xi = 2$.*

Proof. (a) We know that if S is local semi ring with 0 is the only non-semi unit then the ξ is complete (c.f. Theorem 3.1).

Consider $x,y \in S_u = S - \{0\}$, then$x+y \in S_u$ in semi ring the semi units are zero-sum free this implies that there is a path x — y i.e. $d(x,y)=1$. Moreover, there is path x — 0 — y with size 2. But smallest distance $d(x,y)=1$. Hence, $diam\xi = 1$.

(b) Here $|N_u| \geq 2$, so if $a,b \in N_u$ then $a+b \in N_u$. But N_u is k-maximal ideal so $a \in N_u$ and $u \in S_u$, there must $a+u, b+u \in S_u$. So by taking the path a — u — b, $d(a,b)=2$, and therefore $diam\xi = 2$. □

Theorem 3.13. *In Local semi ring S such that $|N_u| \geq 2$*

(a) *If $|S_u| \geq 2$, then $gr(\xi) \in \{3,4\}$*

(b) *If $|S_u| < 2$, then $gr(\xi) = \infty$*

Proof. (a) When $|S_u| \geq 2$, then there exist two or more then two semi units and we have $x+y \in S_u$ and $x+z \in S_u$ for all $x \in N_u$ and $y,z \in S_u$.
If $y+z \in S_u$. So, there is cycle $z \to x \to y \to z$. Hence girth $(\xi)=3$. While if $y+z \notin S_u$, then for any $t \in N_u$, such that $x+t, y+t \in S_u$ then there is cycle $x \to y \to t \to z \to x$ therefore it is cycle of length 4 and girth is greater than 3.

(b) When $|S_u| < 2$, so there is only one semi unit. Consider $x \in N_u$ and $u \in S_u$ so $x+u \in S_u$. Also, for every $x,y \in N_u$ so $x+y \in N_u$. Therefore, there is no cycle. Hence $girth(\xi) = \infty$. □

A graph G is called *multi-graph* if it allows loops and multiple edges otherwise it is called *simple graph*.

Proposition 3.14. *Consider S is semiring with multiplicative identity 1. If S_u is the set of all semi units and $2 \notin S_u$, then after allowing loops and multiple edges on ξ, it remains the simple.*

Proof. S is semiring therefore addition is well defined and there contain no multiple edges. We will show that there is also no loop on any $x \in S$. We have $1+1=2 \notin S_u$, i.e. there is no loop at 1. Now suppose on contrary that $x+x = 2.x \in S_u$. It gives $1+(2.x).r = (2.x).s$, for some $r,s \in S$. This implies that $1+2.(x.r) = 2.(x.s)$ or $1+2.r_1 = 2.s_1$ where $r_1 = x.r$, $s_1 = x.s \in S$ which tells that 2 is semi unit, which is contradiction therefore $2.x \notin S_u$. Hence no loop in whole graph therefore its multiple graph is simple graph. □

Theorem 3.15. *Let S be a semi ring, then the semi unit graph ξ is a closed complete graph if and only if S is semi field with $char(S) = 2$.*

Proof. Suppose that the semi unit graph ξ is a closed complete graph. We will prove that S is semi field with $char(S) = 2$. Let $0 \neq r \in S$, the zero of S is adjacent to r, so every element is semi unit (by definition of semi unit graph) implies that $N_u = \{0\}$.
Before going to show that every non-zero element is unit, we shall prove that $char(S) = 2$. As $0 + 0 = 0 \notin S_u$, therefore there is no loop at 0. The graph is closed complete therefore there would be no loop on any vertex of graph. Hence the graph is simple. This implies that $x + x \notin S_u$, because 0 has no loop so all vertices must have no loops. As $x + x \in N_u = \{0\}$, therefore $2x = 0$, for every $x \in S$. This shows that $charS = 2$.

Now, to check that every r is unit element. We have shown that every non-zero $r \in S$ is semi unit, and $1 + r.a = r.b$ for some $a, b \in S$.

Adding $r.a$ on both sides, we get $1 + 2r.a = r(b + a)$ or $1 = r.(b + a)$, so r is unit element too. Hence S is semi field with $char(S) = 2$.

Conversely, suppose that S is semifield with $char(S) = 2$ therefore every non zero element is a unit so also semi unit and every semi unit must be zero sum free in semi field, that shows for all different $x, y \in S_u$, $x + y \in S_u$. Hence it is complete graph without any loop, as $char(S) = 2$ implies that the possibility of formulation of loop is ruled out so complete closed graph. □

Remark 3.16. Each finite semi field is either a field or isomorphic to Boolean semi field ([13, Corollary 5.9]). Boolean field does not have $Char(2)$ so we take fields as $Char(2)$.

Example 3.17. Let $S = \{0, 1, a, 1 + a\}$, then with following operations, S is field of $Char2$

$+$	0	1	a	$1+a$
0	0	1	a	$1+a$
1	1	0	$1+a$	a
a	a	$1+a$	0	1
$1+a$	$1+a$	a	1	0

$\times$	0	1	a	$1+a$
0	0	0	0	0
1	0	1	a	$1+a$
a	0	a	$1+a$	1
$1+a$	0	$1+a$	1	a

Its graph is complete closed semi unit graph as shown in Figure 1a.

Theorem 3.18. *Let R be a semi ring with multiplicative identity and S is additive group in R with multiplicative identity, then*

(a) *If $2 \notin S_u$, then the semi unit graph ξ of S is $|S_u|$-regular graph.*

(b) *If $2 \in S_u$, then for all $x \in S_u$, we have deg $(x) = |S_u| - 1$ and for all $x \in N_u$ we have $deg(x) = |S_u|$.*

Proof. (a) Suppose $x \in S$ and S is additive group S therefore $S + x = S$. If S_u is the set of all semi units u of S, then there exist some $x_u \in S$ such that $x_u + x = u$. Clearly, x_u is uniquely determined by u. If $2 \notin S_u$, then $x + x = 2.x \notin S_u$. It tells that $x_u \neq x$, so x_u is adjacent in graph with x only, therefore we can define a mapping

$$\theta : S_u \to N_\xi(x),$$

where $N_\xi(x)$ is set of neighborhood vertices of x such that $\theta(u) = x_u$, this function is clearly well defined and bijective therefore $|N_\xi(x)| = |S_u|$, that $deg(x) = |S_u|$, so the graph is $|S_u|$-regular graph.

(b) As S is group under addition therefore $S + x = S$ therefore for every $u \in S_u$, there exists $x_u \in S$ such that $x_u + x = u$. Clearly x_u is uniquely determined by u. Now suppose that $2 \in S_u$ and $x \in N_u$ so that $2.x \in N_u$ (by Theorem 2.5). This tells us that $x_u \neq x$ since $x + x$ is not semiunit so x_u is not adjacent to x, therefore the previous observation of (a) is still valid, which shows that

$$deg(x) = |S_u|.$$

Next suppose that $2 \in S_u$, and $x \in S_u$, then $2.x \in S_u$ (by Proposition 2.2). In this case we have $x_u \neq x$ for $u \neq 2x$ and $x_u = x$ for $u = 2x$. Now define a mapping $\theta : S_u \to N_\xi[x]$ (where $N_\xi[x]$ is set of neighborhood vertices including x itself), such that $\theta(u) = x_u$. This is well-defined and bijective therefore $deg(x) = |N_\xi[x]| - 1$, as loops are not considered in simple graphs. □

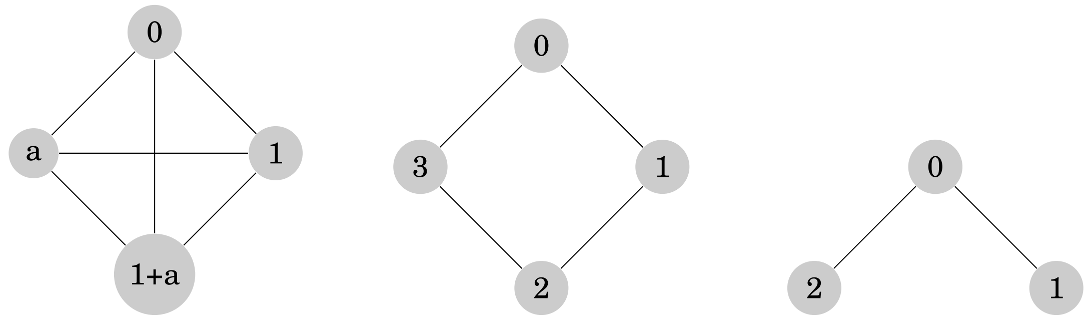
Example 3.19. (1) Consider a semi ring $B(5,1)$,where both operation of addition and multiplication has mod $5 - 1 = 4$. These operations are completely elaborate in the following tables.

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	0	1	2
2	2	3	0	1	2	3
3	3	0	1	2	3	0
4	0	1	2	3	0	1
5	1	2	3	0	1	2

×	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	0	1
2	0	2	0	2	0	2
3	0	3	2	1	0	3
4	0	0	0	0	0	0
5	0	1	2	3	0	1

Here we take out additive group $S = \{0,1,2,3\}$, with identity 1 and $S_u = \{1,3\}$ with $2 \notin S_u$,so its semi unit graph is 2-regular graph as in Figure 1b.

(2) Consider a semi ring $B(5,2)$,where both operation of addition and multiplication has mod $5 - 2 = 3$. Here we take out additive group $S = \{0,1,2\}$, with identity 1 and $S_u = \{1,2\}$ with $2 \in S_u$ then $deg(x) = 2$ if $x \in S_u$ and $deg(x) = 1$ if $x \notin S_u$ as shown in Figure 1c.



(a) (b) (c)

Figure 1

Theorem 3.20. *Let S be a commutative semi ring and M be a Q- maximal ideal of S such that $|S/M| = 2$ and $2 \notin S_u$. Then $\xi(S)$ is complete bipartite graph.*

Proof. Let $V_1 = M$ and $V_2 = S \setminus M$. Here M is Q-ideal therefore M is k-ideal (c.f. [12, Lemma 2]), and $|S/M| = 2$ therefore $S/M = \{M, M + a\}$, for some $a \in Q$, is a semiring with M is zero of S/M. Here M is k-maximal ideal so $N_u \subseteq M = V_1$ implies that $S_u \subseteq M + a = S \setminus M = V_2$. First we show that $N_u = M$. Suppose on contrary that there exist some $x \in N_u \subseteq M$ and $x \in M + a$, implies that $M = M + a$, by definition of Q partitioning, which is a contradiction. Hence $N_u = M$, therefore we have $V(\xi) = V_1 \cup V_2$ and $V_1 \cap V_2 = \phi$, and V_1 and V_2 makes partition of $V(\xi)$ into two subsets.

It is clear that no pair of distinct elements of V_1 are adjacent.Now to prove that $\xi(S)$ is bipartite, we now only to show that no two elements of V_2 are adjacent. Then by hypothesis, there is some $a \in Q$ (a must be semi unit) such that $S = M \cup (M + a)$. Now for distinct, $x, y \in S \setminus M = M + a$. Suppose on contrary that x and y are adjacent then $x + y \in S_u = S \setminus M = M + a$. therefore $x + y = (m_1 + a) + (m_2 + a) = m_1 + m_2 + 2.a = m_3 + 2.a$, we have $2 \notin S_u$, therefore $2a \notin S_u$ hence $x + y = m_3 + 2a = m_4 \in M$. This implies that semiunit $x + y \in M$, which is contradiction, therefore the elements of V_2 are non-adjacent among themselves. Hence $\xi(S)$ is bipartite graph.

Now, we have to prove that $\xi(S)$ is complete bipartite. Let $x \in V_1$ and $y \in V_2$. If $x + y \notin V_2 = S_u = S \setminus M$, then $x + y \in V_1 = M$, and M is Q-ideal therefore k-ideal therefore $y \in M$ which is a contradiction. Thus $x + y \in S_u$ so x and y are adjacent. Therefore each vertex of V_1 is joined to each vertex of V_2. Hence $\xi(S)$ is completely bipartite. □

Example 3.21. Consider a set $S = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, 0\}$, This is semi ring with binary operation addition is max and multiplication is usual multiplication. Here only semi unit is identity 1, while all others are non-semi units. These non-semi units makes maximal ideal, with Q-partitioning. Here $Q = \{1\}$, such that M, $M + 1$ make partitioning, it makes complete bipartite graph.

A semi ring S is Noetherian (respectively, Artinian) if any non-empty set of k-ideals of S has a maximal member (respectively, minimal member) with respect to set inclusion. The definition is equivalent to the ascending chain condition (respectively, descending chain condition) on k-ideals of S. Every finite Semi ring is Noetherian or artinian Semi ring.

Proposition 3.22 ([13]). *Let S be an Artinian cancellative semi ring. Then*

(i) *Every element of S is either a semi unit or a nilpotent element.*

(ii) *S is a local semi ring.*

The following theorem is straight forward from Proposition 3.15 and previous results.

Theorem 3.23. (a) *Let S be an Artinian cancellative semi ring with identity then is connected.*

(b) *Let S be an Artinian cancellative semi ring, then for any non-semi unit x, deg (x) = $|S_u|$.*

(c) *Let S be an Artinian cancellative semi ring then for all semi units x, $deg(x) = |S| - 1$.*

(d) *Let S be an Artinian cancellative semi ring then $d(a, b) = 1$ or 2, for all $a, b \in S$.*

(e) *Let S be an Artinian cancellative semi ring then $diam(\xi) \leq 2$.*

Example 3.24. Consider an artinian semi ring $Z^+ \cup \{0,\infty\}$. Here $S_u = \{1,2,3,\dots,\infty\}$ and $N_u = \{0\}$, we can easily check all the above mentioned results of Theorem 3.23 are valid.

Let G be a connected graph. The minimum number of vertices whose removal makes G either disconnected or reduces G in to a trivial graph is called its vertex connectivity, denoted by $k(G)$

Theorem 3.25. *If S is local semi ring then ξ is connected then $k(\xi) = |S_u|$.*

Proof. Consider S is local semi ring then non-semi units N_u make k-maximal ideal (by Proposition 3.3), and for every $x \in N_u$ and $u \in S_u$ implies that $x + u \in S_u$. So, there is connection between every non-semi unit and unit. To make the graph disconnected we shall disconnect semiunits from non-semiunits i.e. there must not edge between semiunits with non-semi units, so $k(\xi) = |S_u|$. □

Corollary 3.26. *If S is Artinian semi ring then ξ is connected, while $k(\xi) = |S_u|$.*

Corollary 3.27. *If S is finite local semi ring and $S_u = 1$, then semi unit graph ξ is connected, complete with $k(\xi) = |S_u| = 1$.*

Gelfand semiring is defined in [10]. *It is a semiring S with identity* 1*, such that $1 + a$ is a unit for all $a \in S$. In Gelfand semiring the sum of two units is always a unit* [9].

Lemma 3.28. *In a Gelfand semi ring S ($|S| \geq 3$) with non-zero identity along with unit elements ($U(S) \geq 2$) then the girth of semi unit graph is* 3.

Proof. If $a, b \in S$ then $1 + a, 1 + b \in S_u$. Also, the sum of two units in a Gelfand semi ring is also unit ([10, Proposition 4.50]) so this will create a cycle $1 — 1 + a — 1 + b — 1$. Hence the girth of its semiunit graph is 3. □

A cycle that passes through every vertex in a graph is called a Hamilton cycle and a graph with such a cycle is called Hamiltonian. Ore's theorem states that if $deg(x) + deg(y) \geq n$ for every pair of distinct non-adjacent vertices x and y of graph G of n vertices ($n \geq 3$) then G is Hamiltonian [14]. The following theorem investigate about Semiunit graph to be Hamiltonian.

Theorem 3.29. *In a local semi ring S, if $|S| \geq 3$ with $|S_u| = |N_u|$, then $\xi(S)$ is Hamiltonian graph.*

Proof. Consider S to be local semiring then N_u is a k-ideal and so every semi unit is adjacent with every non semi unit. For any $x \in N_u$ the degree $d(x) = |S_u| = |N_u|$, so if there exist non-adjacent vertices then for any non-adjacent vertices $x, y \in N_u$,

$$d(x) + d(y) = |S_u| + |S_u| = |S| \tag{3.1}$$

also, if $x, y \in S_u$, then

$$d(x) + d(y) \geq |N_u| + |N_u| = |N_u| + |N_u| = |S|. \tag{3.2}$$

From (3.1) and (3.2), we have $d(x) + d(y) \geq |S|$, therefore this makes Hamiltonian Graph. □

Conclusion

The research in this area aims at exploring the relationship between the semirings and graph theory. Here, we discuss semi unit elements and their graphs which enables us to characterize the semirings in terms of semi units and non-semi units. In this paper, we have defined semi unit graphs $\xi(S)$ and have discussed some basic properties. Further, we have studies the connectedness, diameter, girth, completeness and connectivity of $\xi(S)$. These graphs pave the way for the further research to find chromatic and clique number of semi unit graph and to find the applications in Biology, Chemistry and Computer sciences.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] P. J. Allen, A fundamental theorem of homomorphisms for semirings, *Proc. Amer. Math. Soc.* **21** (1969), 412 – 416, DOI: 10.1090/S0002-9939-1969-0237575-4.

[2] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative rings, *J. Algebra* **217** (1999), 434 – 447, DOI: 10.1006/jabr.1998.7840.

[3] S. E. Atani, An ideal based zero divisor graph of commutative rings, *Glasnik Math.* **44** (2009), 141 – 153, DOI: 10.3336/gm.44.1.07.

[4] S. E. Atani, The ideal theory in quotients of commutative semirings, *Glasnik Math. Ser. III* 42 (2007), 301 – 308, DOI: 10.3336/gm.42.2.05.

[5] S. E. Atani, The zero-divisor graph with respect to ideals of a commutative semirings, *Glasnik Math.* **43** (2008), 309 – 320, DOI: 10.3336/gm.43.2.06.

[6] S. E. Atani and R. Atani, Some remarks on partitioning semirings, *An. St. Univ. Ovidius Constanta* **18** (2010), 49 – 62, `https://www.academia.edu/487767/`.

[7] R. E. Atani and S. E. Atani, Ideal theory in commutative semirings, *Bul. Acad. Stiinte Repub. Mold. Mat.* **2** (2008), 14 – 23, `http://www.mathnet.ru/php/archive.phtml?wshow=paper&jrnid=basm&paperid=15&option_lang=eng`.

[8] I. Beck, Coloring of a commutative ring, *J. Algebra* **116** (1988), 208 – 226, DOI: 10.1016/0021-8693(88)90202-5.

[9] J. S. Golan, *Semi rings and Affine Equations over Them: Theory and Applications*, Springer Science + Business Media, Dordrecht (2003), DOI: 10.1007/978-94-017-0383-3.

[10] J. S. Golan, *Semirings and their Applications*, Kluwer Academic Pub., Dordrecht (1999), DOI: 10.1007/978-94-015-9333-5.

[11] J. S. Golan, The theory of semirings with applications in mathematics and theoretical computer Science, *Pitman Monographs and Surveys in Pure and Applied Mathematics*, Longman Scientific and Technical (1992), `https://catalog.princeton.edu/catalog/724763`.

[12] V. Gupta and J. N. Chaudhari, Right pi-regular semirings, *Sarajevo J. Math.* **2** (2006), 3 – 9, http://www.anubih.ba/Journals/vol-2,no-1,y06/02vishnugupta1.pdf.

[13] U. Hebisch and H. J. Weinert, *Semirings: Algebraic Theory and Applications in Computer Science*, Series in Algebra: Vol. 5, World Scientific Publishing Co. Ltd., Singapore (1993), DOI: 10.1142/3903.

[14] O. Ore, Note on Hamilton circuits, *American Mathematical Monthly* **67**(1) (1960), 55, DOI: 10.2307/2308928.

[15] M. K. Sen and M. R. Adhikari, On maximal k-ideals of simirings, *Proc. Amer. Math. Soc.* **118** (1993), 699 – 703, DOI: 10.1090/S0002-9939-1993-1132423-6.

[16] W. D. Wallis, *A Beginner's Guide to Graph Theory*, 2nd edition, Springer (2007), DOI: 10.1007/978-0-8176-4580-9.