A New Type of Ideal Convergence of Difference Sequence in Probabilistic Normed Space

Vakeel A. Khan\(^1\),\(^*\), Henna Altaf\(^1\) and Mohammad Faisal Khan\(^2\)

\(^1\)Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
\(^2\)College of Science and Theoretical Studies, Saudi Electronic University, Riyadh, 11673, Saudi Arabia

\(^*\)Corresponding author: vakhanmaths@gmail.com

Abstract. The idea of difference sequence sets \(X(\Delta) = \{x = (x_k) : \Delta x \in X\}\) with \(X = l_\infty, c\) and \(c_0\) was introduced by Kizmaz [10]. Mursaleen and Mohiuddine [13] defined the idea of probabilistic normed space (PNS) and the ideal convergence in PNS. Motivated by the above two concepts, we in this paper introduce the notion of difference \(I\)-convergent sequence in PNS and study the elementary properties of this convergence.

Keywords. Triangular norm; Probabilistic normed space; \(\Delta I\)-convergence; \(\Delta I^*\)-convergence; \(\Delta I\)-limit points; \(\Delta I\)-cluster points

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1. Introduction

Statistical convergence for real sequences was defined by Fast [4] and Steinhaus [17] in 1951 and this concept was studied and applied by many authors, namely [6], [3]. This idea was generalised to \(I\)-convergence by Kostyrko et al. [11]. Lately, \(I\)-convergence for sequence of functions has been studied by Balcerzak et al. [2].
The idea of probabilistic normed spaces is the generalisation of normed space and has emerged from the concept of statistical metric spaces which was defined by Menger [12] and later on was studied by Schweizer and Sklar [16]. It is useful in many areas like continuity properties [1], topological spaces [5] etc. In 2012, M. Mursaleen and S.A. Mohiuddine gave the concept of ideal convergence in probabilistic normed space (PNS) [13]. Difference sequence spaces \( l_\infty(\Delta), c(\Delta) \) and \( c_0(\Delta) \) were defined by H. Kizmaz [10].

In this paper, we have defined the notion of difference ideal convergence of sequences in probabilistic normed space. Section 2 deals with the idea of \( \Delta I \)-and \( \Delta I^* \)-PNS and some basic algebraic properties of these notions. In section 3, we have defined \( \Delta I \)-limit points and \( \Delta I \)-cluster points in PNS and studied some related results.

To recall certain definitions such as ideal, \( I \)-convergence, solid space, sequence algebra, difference convergence in probabilistic normed space. etc which will be used throughout this paper, we refer to [11], [7], [9], [15], [8], [14], [14], [18]. We denote by \( \mathbb{N}, \mathbb{C}, \mathbb{R}, \mathbb{R}^*_+ \) as the set of natural numbers, complex numbers, real numbers and positive real numbers.

2. Main Results

In this section, we study the concept \( \Delta I \)-and \( \Delta I^* \)-convergence of sequence in probabilistic normed space. We take ideal \( I \) as non-trivial admissible ideal.

2.1 \( \Delta I \)-Convergence in PNS

**Definition 2.1.** Consider an ideal \( I \) in \( \mathbb{N} \) and \( (X, \nu, \ast) \) be a probabilistic normed space. A sequence \( x = (x_k) \in X \) is said to be \( \Delta I \)-convergent to \( l \in X \) with respect to the probabilistic norm \( \nu \) if for each \( \epsilon > 0 \) and \( t > 0 \)

\[
\{ k \in \mathbb{N} : \nu_{\Delta x_k - l}(t) \leq 1 - \epsilon \} \in I.
\]

We write \( I_{\nu \Delta} \lim x = l \) and the space of all such sequences as \( \Delta(c^I) \).

**Theorem 2.1.** Let \( (X, \nu, \ast) \) be a probabilistic normed space. Then the following conditions are equivalent:

(a) \( I_{\nu \Delta} \lim x = l \).

(b) \( \{ k \in \mathbb{N} : \nu_{\Delta x_k - l}(t) \leq 1 - \epsilon \} \in I_{\nu} \) for each \( \epsilon > 0 \) and \( t > 0 \).

(c) \( \{ k \in \mathbb{N} : \nu_{\Delta x_k - l}(t) > 1 - \epsilon \} \in \mathfrak{P}(I_{\nu}) \) for each \( \epsilon > 0 \) and \( t > 0 \).

(d) \( I_{\ast} \lim \nu_{\Delta x_k - l}(t) = 1 \).

**Proof.** The proof follows from Definition 2.1. \( \square \)

**Theorem 2.2.** Let \( (X, \nu, \ast) \) be a PNS. If a sequence \( x = (x_k) \) is \( \Delta I \)-convergent then \( I_{\nu \Delta} \)-limit is unique.

**Proof.** Suppose there are two limits \( l_1 \) and \( l_2 \) with \( l_1 \neq l_2 \). Given \( r > 0 \) such that \( (1 - r) \ast (1 - r) \geq 1 - \epsilon \) for \( \epsilon > 0 \). Define the sets for \( t > 0 \) as follows:

\[
A_{\nu, 1}(r, t) = \{ k \in \mathbb{N} : \nu_{\Delta x_k - l_1}(t) \leq 1 - r \},
\]

\[
A_{\nu, 2}(r, t) = \{ k \in \mathbb{N} : \nu_{\Delta x_k - l_2}(t) \leq 1 - r \}.
\]
We prove Theorem 2.3. Let \( \alpha \) thus \( I \). Then by definition of \( \Delta I \)-convergence \( A_{\nu,1}(r,t) \) and \( B_{\nu,2}(r,t) \in I \) and hence \( C_{\nu}(r,t) = A_{\nu,1}(r,t) \cup B_{\nu,2}(r,t) \in I \). This implies \( C_{\nu}(r,t)^c \in \mathcal{F}(I) \) so is non-empty. Let \( n \in C_{\nu}(r,t)^c \) then \( n \in A_{\nu,1}(r,t)^c \cap B_{\nu,2}(r,t)^c \). So,

\[
v_{l_1-l_2}(t) \geq v_{\Delta x_{-l_1}} \left( \frac{t}{2} \right) \ast v_{\Delta x_{-l_2}} \left( \frac{t}{2} \right) > (1-r) \ast (1-r) \geq 1-\epsilon.
\]

It follows that \( v_{l_1-l_2} > 1-\epsilon \). Since \( \epsilon \) is arbitrary, we have \( v_{l_1-l_2}(t) = 1 \Rightarrow l_1 = l_2 \).

This completes the proof. \( \square \)

**Theorem 2.3.** Let \((X, \nu, *)\) be a probabilistic normed space. Then

(a) If \( \nu_{\Delta}\)-lim \( x = l \), then \( I_{\nu_{\Delta}}\)-lim \( x = l \).

(b) If \( I_{\nu_{\Delta}}\)-lim \( x = l_1 \) and \( I_{\nu_{\Delta}}\)-lim \( y = l_2 \), then \( I_{\nu_{\Delta}}\)-lim \( (x+y) = l_1 + l_2 \).

(c) If \( I_{\nu_{\Delta}}\)-lim \( x = l \), then \( I_{\nu_{\Delta}}\)-ax = al.

**Proof.** (a): Suppose \( \nu_{\Delta}\)-lim \( x = l \), then by definition for each \( \epsilon > 0 \) and \( t > 0 \) there exists \( N > 0 \) such that

\[
v_{\Delta x_{-l}}(t) > 1-\epsilon \quad \text{for each} \quad k > N.
\]

Observe that \( A(t) = \{ k \in \mathbb{N} : v_{\Delta x_{-l}}(t) \leq 1 - \epsilon \} \subseteq \{1, 2, 3, \ldots, N-1 \} \). Since \( I \) is admissible ideal, therefore \( A(t) \in I \). Hence \( I_{\nu_{\Delta}}\)-lim \( x = l \).

(b): Let \( I_{\nu_{\Delta}}\)-lim \( x = l_1 \) and \( I_{\nu_{\Delta}}\)-lim \( y = l_2 \). For given \( \epsilon > 0 \) and \( t > 0 \) given \( r > 0 \) with \( (1-r) \ast (1-r) > 1-\epsilon \). By definition the sets

\[
A_{\nu,1}(r,t) = \{ k \in \mathbb{N} : v_{\Delta x_{-l}}(t) \leq 1-r \} \in I,
\]

\[
B_{\nu,2}(r,t) = \{ k \in \mathbb{N} : v_{\Delta y_{-l}}(t) \leq 1-r \} \in I.
\]

\( C_{\nu}(r,t) = A_{\nu,1}(r,t) \cup B_{\nu,2}(r,t) \in I \) so that \( C_{\nu}(r,t)^c \in \mathcal{F}(I) \). We prove

\[
C_{\nu}(r,t)^c \subseteq \{ k \in \mathbb{N} : v_{\Delta(x+y)-(l_1+l_2)}(t) > 1-\epsilon \}.
\]

Let \( k \in C_{\nu}(r,t)^c \), then

\[
v_{\Delta x_{-l_1} + \Delta y_{-l_2}}(t) \geq v_{\Delta x_{-l_1}} \left( \frac{t}{2} \right) \ast v_{\Delta y_{-l_2}} \left( \frac{t}{2} \right) > (1-r) \ast (1-r) > 1-\epsilon.
\]

Therefore,

\[
C_{\nu}(r,t)^c \subseteq \{ k \in \mathbb{N} : v_{\Delta x_{-l_1} + \Delta y_{-l_2}}(t) > 1-\epsilon \}.
\]

Hence

\[
\{ k \in \mathbb{N} : v_{\Delta x_{-l_1} + \Delta y_{-l_2}}(t) > 1-\epsilon \} \in I.
\]

Thus \( I_{\nu_{\Delta}}\)-lim \( x_{k} + y_{k} = l_1 + l_2 \).

(c): The proof holds for \( \alpha = 0 \). Let \( \alpha \neq 0 \). We are given \( I_{\nu_{\Delta}}\)-lim \( x = l \), therefore the set

\( A(t) = \{ k \in \mathbb{N} : v_{\Delta x_{-l}}(t) > 1-\epsilon \} \in \mathcal{F}(I) \).

We prove \( A(t) \subseteq \{ k \in \mathbb{N} : v_{\Delta x_{-al}}(t) > 1-\epsilon \} \). Let \( k \in A(t) \). Then by definition \( v_{\Delta x_{-l}}(t) > 1-\epsilon \).
Now
\[ v_{\Delta ax_{-at}}(t) = v_{\Delta x_{-at}} \left( \frac{t}{|a|} \right) \geq v_{\Delta x_{-l}}(t) \ast v_0 \left( \frac{t}{|a|} - t \right) = v_{\Delta x_{-l}}(t) \ast 1 = v_{\Delta x_{-l}}(t) > 1 - \epsilon. \]

Hence, we have \[ A(t) \subseteq \{ k \in \mathbb{N} : v_{\Delta ax_{-at}}(t) > 1 - \epsilon \} \] and therefore \[ I_{\nu\Delta}\lim ax = al. \]

### 2.2 \( \Delta I^* \)-Convergence in PNS

In this section, we introduce the concept of \( \Delta I^* \)-convergence of sequences in probabilistic normed space.

**Definition 2.2.** Consider PNS \((X, \nu, \ast)\). A sequence \( x = (x_k) \in X \) is said to be \( \Delta I^* \)-convergent to \( l \in X \) with respect to the probabilistic norm \( \nu \) if there exists \( K = \{ k_m : k_1 < k_2 < \cdots \} \subseteq \mathbb{N} \) such that \( K \in \mathcal{F}(I) \) and \( \nu_{\Delta}\lim x_{k_m} = l \). We write \( I_{\nu\Delta}^*\lim x = l \).

**Theorem 2.4.** Let \((X, \nu, \ast)\) be a PNS and \( I \) be an admissible ideal. If \( I_{\nu\Delta}^*\lim x = l \) then \( I_{\nu\Delta}\lim x = l \).

**Proof.** Let \( I_{\nu\Delta}^*\lim x = l \). Then by definition there exists \( K = \{ k_m : k_1 < k_2 < \cdots \} \in \mathcal{F}(I) \) \((K^c = H\text{(say)} \in \mathcal{I})\) and \( \nu_{\Delta}\lim x_{k_m} = l \). Then for each \( \epsilon > 0 \) and \( t > 0 \) there exists \( N > 0 \) such that \( v_{\Delta x_{k_m}}(t) > 1 - \epsilon \) for all \( m > N \). Since \( \{ k_m \in K : v_{\Delta x_{k_m}}(t) > 1 - \epsilon \} \subseteq \{ k_1 < k_2 < \cdots < k_{N-1} \} \) and \( I \) is an admissible ideal, we have
\[
\{ k_m \in K : v_{\Delta x_{k_m}}(t) > 1 - \epsilon \} \subseteq I.
\]

Hence
\[
\{ k \in \mathbb{N} : v_{\Delta x_k}(t) > 1 - \epsilon \} \subseteq H \cup \{ k_1 < k_2 < \cdots < k_{N-1} \} \subseteq I
\]
for each \( \epsilon > 0 \) and \( t > 0 \). It follows \( I_{\nu\Delta}\lim x = l \). \( \square \)

**Remark 2.1.** The converse of above theorem is not necessarily true which is shown by the below given example.

**Example 2.1.** Consider the normed space \((\mathbb{R}, | \cdot |)\) with the usual norm and let \( a \ast b = ab \) for all \( a, b \in [0, 1] \). Define
\[
v_x(t) := \frac{t}{t + |x|} \quad \text{for all } x \in \mathbb{R} \text{ and every } t > 0.
\]

Then \((\mathbb{R}, v, \ast)\) is a PNS. Let \( \mathbb{N} = \bigcup_j D_j \) be a decomposition of \( \mathbb{N} \) such that for any \( n \in \mathbb{N} \) each \( D_j \) contains infinitely many \( j \)'s where \( j \geq n \) and \( D_j \cap D_n = \phi \). Let \( I \) be the class of all subsets of \( \mathbb{N} \) which intersects with at most a finite number of \( D_j \)'s, then \( I \) is an admissible ideal. Define a sequence \( x_n = \frac{1}{j} \) if \( n \in D_j \). Then
\[
v_{\Delta x_n}(t) = \frac{t}{t + |\Delta x_n|} \to 1 \quad \text{as} \quad n \to \infty.
\]

Hence \( I_{\nu\Delta}\lim x_n = 0 \). Now suppose that \( I_{\nu\Delta}^*\lim x_n = 0 \), then there exists \( K = \{ n_j : n_1 < n_2 < \cdots \} \subseteq \mathbb{N} \) with \( K \in \mathcal{F}(I) \) and \( \nu_{\Delta}\lim x_{n_j} = 0 \). Since \( K \in \mathcal{F}(I) \), we have \( K^c = H\text{(say)} \in \mathcal{I} \). Then, there exists \( p \in \mathbb{N} \) such that \( H \subseteq \bigcup_{n=1}^p D_n \).
Theorem 2.5. Let \((X, \nu, *)\) be a PNS and \(I\) satisfies AP condition. Then \(I_{\nu^*}\)-lim \(x = l\) implies \(I_{\nu^*}\)-lim \(x = l\).

Proof. Suppose \(I\) be an admissible ideal that satisfies AP condition and \(I_{\nu^*}\)-lim \(x = \xi\). By definition for each \(\epsilon > 0\) and \(t > 0\), we have \(\{k \in \mathbb{N} : \nu_{\Delta x_k - \xi}(t) \leq 1 - \epsilon\} \in I\). Define the set \(A_p\) for \(p \in \mathbb{N}\)

\[
A_p = \left\{k \in \mathbb{N} : 1 - \frac{1}{p} \leq \nu_{\Delta x_k - \xi}(t) < 1 - \frac{1}{p+1}\right\}.
\]

Observe that \(\{A_1, A_2, \cdots\}\) is a countable set, belongs to \(I\) and \(A_i \cap A_j = \phi\) for \(i \neq j\). There exists a countable family of sets \(\{B_1, B_2, \cdots\} \in I\) such that the symmetric difference \(A_i \Delta B_i\) is a finite set for each \(i \in \mathbb{N}\) and \(B = \bigcup_{i=1}^{\infty} B_i \in I\). Hence \(B^c = K\) (say) \(\in \mathcal{F}(I)\). We will prove \((x_k)_{k \in K}\) is \(\Delta\nu\)-convergent to \(\xi\). Let \(\eta > 0\) and \(t > 0\), choose \(q \in \mathbb{N}\) such that \(\frac{1}{q} < \eta\). Then

\[
\{k \in \mathbb{N} : \nu_{\Delta x_k - \xi}(t) \leq 1 - \eta\} \subset \left\{k \in \mathbb{N} : \nu_{\Delta x_k - \xi}(t) \leq 1 - \frac{1}{q}\right\} \subset \bigcup_{i=1}^{q+1} A_i.
\]

Since \(A_i \Delta B_i\), \(i = 1, 2, \cdots, q + 1\) are finite, there exists \(k_0 \in \mathbb{N}\) such that

\[
\bigcup_{i=1}^{q+1} B_i \cap \{k : k \geq k_0\} = \bigcup_{i=1}^{q+1} A_i \cap \{k : k \geq k_0\}.
\]

If \(k \geq k_0\) and \(k \in K\) then \(k \not\in \bigcup_{i=1}^{q+1} B_i\). Therefore, \(k \not\in \bigcup_{i=1}^{q+1} A_i\). Hence for every \(k \geq k_0\) and \(k \in K\), we have \(\nu_{\Delta x_k - \xi}(t) > 1 - \eta\).

Since \(\eta > 0\) is arbitrary, we have \(I_{\nu^*}\)-lim \(x = \xi\). \(\Box\)

Theorem 2.6. Let \((X, \nu, *)\) be a PNS. Then the following statements are equivalent:

(i) \(I_{\nu^*}\)-lim \(x = l\).

(ii) There exist two sequences \(y = (y_k)\) and \(z = (z_k)\) in \(X\) such that \(x = y + z\), \(\nu_{\Delta}\)-lim \(y = \xi\) and the set \(\{k : z_k \neq 0\} \in I\), where \(\theta\) denotes the zero element of \(X\).

Proof. Suppose (i) holds. Then there exists \(K = \{k_m : k_1 < k_2 < \cdots\} \subseteq \mathbb{N}\) such that \(K \in \mathcal{F}(I)\) and \(\nu_{\Delta}\)-lim \(x_{k_m} = l\). Define the sequences \(y\) and \(z\) as follows:

\[
y_k = \begin{cases} x_k & \text{if } k \in K \\ l & \text{if } k \in K^c \end{cases}
\]

and \(z_k = x_k - y_k\) for all \(k \in \mathbb{N}\). For each \(\epsilon > 0\), \(t > 0\) and \(k \in K^c\), we have \(\nu_{\Delta y_k - l}(t) = 1 > 1 - \epsilon\). Thus \(\nu_{\Delta}\)-lim \(y = l\). Since \(\{k : z_k \neq 0\} \in K^c\), we have \(\{k : z_k \neq 0\} \in I\).

Let (ii) holds. Then the set \(K = \{k : z_k \neq 0\} \in \mathcal{F}(I)\) is an infinite set. Let \(K = \{k_m : k_1 < k_2 < \cdots\}\). Since \(x_{k_m} = y_{k_m}\) and \(\nu_{\Delta}\)-lim \(y = l\), \(\nu_{\Delta}\)-lim \(x_{k_m} = l\). Therefore \(I_{\nu^*}\)-lim \(x = l\). \(\Box\)
3. \( \triangle I \)-Limit Points and \( \triangle I \)-Cluster Points in PNS

Definition 3.1. Let \((X, \nu, \ast)\) be a probabilistic normed space, \(l \in X\) is said to be \(\triangle\)-limit point of sequence \(x = (x_k) \in X\) with respect to the probability norm \(\nu\) if there is a subsequence of \(x\) that \(\triangle\)-converges to \(l\) with respect to the probabilistic norm \(\nu\). By \(\mathcal{L}_{\nu, \triangle}(x)\), we denote the set of all \(\triangle\)-limit points of the sequence \(x\).

Definition 3.2. Let \((X, \nu, \ast)\) be a PNS. An element \(l \in X\) is said to be \(\triangle I\)-limit point of the sequence \(x = (x_k) \in X\) with respect to the probabilistic norm \(\nu(I_{\nu, \triangle}\)-limit point) if there is a subset \(K = \{k_1 < k_2 < \cdots \} \subseteq \mathbb{N}\) such that \(K \in I\) and \(\nu_{\triangle}(x_{k_m}) = l\). We denote by \(\Lambda_{\nu, \triangle}^I(x)\), the set of all \(I_{\nu, \triangle}\)-limit points of the sequence \(x = (x_k)\).

Definition 3.3. Let \((X, \nu, \ast)\) be a PNS. An element \(l \in X\) is said to be \(\triangle I\)-cluster point of \(x = (x_k) \in X\) with respect to the probabilistic norm \(\nu\) if for each \(\epsilon > 0\) and \(t > 0\)
\[
K = \{k \in \mathbb{N} : \nu_{\triangle}(x_{k_m}) > t + \epsilon\} \notin I.
\]
We denote by \(\Gamma_{\nu, \triangle}^I(x)\) the set of all \(I_{\nu, \triangle}\)-cluster points of the sequence \(x\).

Theorem 3.1. Let \((X, \nu, \ast)\) be a PNS. Then \(\Lambda_{\nu, \triangle}^I(x) \subseteq \Gamma_{\nu, \triangle}^I(x) \subseteq \mathcal{L}_{\nu, \triangle}(x)\), where \(x = (x_k) \in X\).

Proof. Let \(l \in \Lambda_{\nu, \triangle}^I(x)\), then there exists \(K = \{k_n : k_1 < k_2 < \cdots \} \subseteq \mathbb{N}\) such that \(K \notin I\) and \(\nu_{\triangle}(x_{k_m}) = l\). For each \(\epsilon > 0\) and \(t > 0\), there exists \(N \in \mathbb{N}\) such that for \(k > N\), we have \(\nu_{\triangle}(x_{k_m}) > t + \epsilon\). So,
\[
\{k \in \mathbb{N} : \nu_{\triangle}(x_{k_m}) > t + \epsilon\} \supseteq \{k_{N+1}, k_{N+2}, \cdots \}
\]
and thus \(\{k \in \mathbb{N} : \nu_{\triangle}(x_{k_m}) > t + \epsilon\} \notin I\) which implies \(l \in \Gamma_{\nu, \triangle}^I(x)\).

Let \(l \in \Gamma_{\nu, \triangle}^I(x)\) then for each \(\epsilon > 0\) and \(t > 0\), we have \(\{k \in \mathbb{N} : \nu_{\triangle}(x_{k_m}) > t + \epsilon\} \notin I\). Let \(K = \{k_m : k_1 < k_2 < \cdots \}\). Then there is a subsequence \((x_{k_m})\) of \((x_n)\) that \(\triangle\)-converges to \(l\) with respect to the probabilistic norm \(\nu\). Hence \(l \in \mathcal{L}_{\nu, \triangle}(x)\). \(\Box\)

Theorem 3.2. Let \((X, \nu, \ast)\) be a PNS and \(x = (x_k)\) be a sequence in \(X\). Then \(\Lambda_{\nu, \triangle}^I(x) = \Gamma_{\nu, \triangle}^I(x) = \{l\}\), if \(I_{\nu, \triangle}\)-lim \(x = l\).

Proof. Let \(l_1, l_2 \in \Lambda_{\nu, \triangle}^I(x)\), with \(l_1 \neq l_2\). Then there exist two subsets \(K, K'\), \(K = \{k_m : k_1 < k_2 < \cdots \}\) and \(K' = \{p_m : p_1 < p_2 < \cdots \} \subseteq \mathbb{N}\) such that
\[
K \notin I\quad \text{and} \quad \nu_{\triangle}(x_{k_m}) = l_1
\]
\[
K' \notin I\quad \text{and} \quad \nu_{\triangle}(x_{p_m}) = l_2.
\]
Given \(\epsilon > 0\) and \(t > 0\) there exists \(N \in \mathbb{N}\) with \(m > N\), we have \(\nu_{\triangle}(x_{p_m}) > t + \epsilon\). Hence
\[
A = \{p_m \in K' : \nu_{\triangle}(x_{p_m}) > t + \epsilon\} \subseteq \{p_m : p_1 < p_2 < \cdots \} \supseteq \mathbb{N}.
\]
As \(I\) is an admissible ideal so \(A \in I\). If \(B = \{p_m \in K' : \nu_{\triangle}(x_{p_m}) > t + \epsilon\}\), then \(B \notin I\).

Otherwise if \(B \in I\), then \(A \cup B = K' \in I\) which is a contradiction. Also \(I_{\nu, \triangle}\)-lim \(x = l_1\), we have
\[
C = \{k \in \mathbb{N} : \nu_{\triangle}(x_{k_m}) > t + \epsilon\} \subseteq \mathbb{N}.
\]
Hence
\[
C^c = \{k \in \mathbb{N} : \nu_{\triangle}(x_{k_m}) > t + \epsilon\} \subseteq \mathbb{N}.
\]
Since for every \( l_1 \neq l_2, B \cap C = \phi, B \subset C \). Hence \( \Lambda^I_{v\Delta}(x) = l_1 \). On the other hand, suppose \( l_1, l_2 \in \Gamma^I_{v\Delta}(x) \), with \( l_1 \neq l_2 \). Then for each \( \epsilon > 0 \) and \( t > 0 \), we have

\[
A = \{ k \in \mathbb{N} : v_{\Delta x_k - l_1}(t) > 1 - \epsilon \} \notin I,
\]

\[
B = \{ k \in \mathbb{N} : v_{\Delta x_k - l_2}(t) > 1 - \epsilon \} \notin I.
\]

Observe that \( A \cap B = \phi \) and therefore \( B \subset A^c \). Also, \( I_{v\Delta}\)-lim \( x = l_1 \) implies \( A^c = \{ k \in \mathbb{N} : v_{\Delta x_k - l_1}(t) \leq 1 - \epsilon \} \subset I \). Hence \( B \in I \), which is a contradiction. Therefore, \( \Gamma^I_{v\Delta}(x) = l_1 \).

**Theorem 3.3.** Let \((X, \nu, \ast)\) be a PNS. If \( x = (x_k) \) and \( y = (y_k) \) be two sequences in \( X \) and \( A = \{ k \in \mathbb{N} : x_k \neq y_k \} \notin I \). Then \( \Lambda^I_{v\Delta}(x) = \Lambda^I_{v\Delta}(y) \) and \( \Gamma^I_{v\Delta}(x) = \Gamma^I_{v\Delta}(y) \).

**Proof.** Suppose \( l \in \Lambda^I_{v\Delta}(x) \), then by definition there exists \( K = \{ k_m : k_1 < k_2 < \cdots \} \subseteq \mathbb{N} \) such that \( K \notin I \) and \( v_{\Delta \lim x_{k_m}} = l \). For each \( \epsilon > 0 \) and \( t > 0 \), we can find \( N \in \mathbb{N} \) such that \( v_{\Delta x_k - l}(t) > 1 - \epsilon \) for \( m > N \). Define \( K_1 = K \cap A \) and \( K_2 = K \setminus A \). \( A \in I \) implies \( K_1 \in I \). As \( K = K_1 \cup K_2 \) and \( K \notin I \) so \( K_2 \notin I \). It is obvious that the subsequence \( (y_k)_{k \in K_2} \) of the sequence \( y = (y_k) \) is \( \Delta \nu \)-convergent to \( l \). Hence \( l \in \Lambda^I_{v\Delta}(y) \) and therefore \( \Lambda^I_{v\Delta}(x) \subseteq \Lambda^I_{v\Delta}(y) \). Similarly, we can prove \( \Lambda^I_{v\Delta}(y) \subseteq \Lambda^I_{v\Delta}(x) \). Thus \( \Lambda^I_{v\Delta}(x) = \Lambda^I_{v\Delta}(y) \). Let \( l \in \Gamma^I_{v\Delta}(x) \), then for each \( \epsilon > 0 \) and \( t > 0 \), we have

\[
B = \{ k \in \mathbb{N} : v_{\Delta y_k - l}(t) > 1 - \epsilon \} \notin I.
\]

Define \( C = \{ k \in \mathbb{N} : v_{\Delta y_k - l}(t) > 1 - \epsilon \} \). We need to show \( C \notin I \). Suppose on contrary \( C \in I \) then \( C^c \in \mathcal{F}(I) \). Also we have \( A_c \in \mathcal{F}(I) \). Thus \( C_c \cap A_c \in \mathcal{F}(I) \). \( C_c \cap A_c \subset B^c \) implies \( B^c \in \mathcal{F}(I) \). Hence \( B \in I \) which is a contradiction. So \( C \notin I \) and \( \Gamma^I_{v\Delta}(x) \subseteq \Gamma^I_{v\Delta}(y) \). Similarly \( \Gamma^I_{v\Delta}(y) \subseteq \Gamma^I_{v\Delta}(x) \) and hence \( \Gamma^I_{v\Delta}(x) = \Gamma^I_{v\Delta}(y) \).

**4. Conclusion**

In the present paper we have introduced and studied the notion of difference \( I \)-convergence of sequence in PNS and elementary properties of this convergence. We investigated the general type of \( I \)-convergence for difference sequences, that is, Difference Ideal Convergence in Probabilistic Normed Spaces in more general setting. These definitions and results provide new tools to deal with the convergence problems of sequences occurring in many branches of science and engineering.

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Competing Interests

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Authors’ Contributions

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