IF-dimension of Modules

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Abstract. In this paper, we introduce a dimension, called IF-dimension, for modules. With this new dimension we give a new characterization of IFD ring dimension of rings; see [4]. The relations between the IF dimension and other dimensions are discussed.

1. Introduction

Throughout this paper, $R$ denotes a non-trivial associative ring and all modules — if not specified otherwise — are left and unitary.

Let $R$ be a ring, and let $M$ be an $R$-module. As usual we use $\text{pd}_R(M)$, $\text{id}_R(M)$ and $\text{fd}_R(M)$ to denote, respectively, the classical projective dimension, injective dimension and flat dimension of $M$. We use also $\text{gldim}(R)$ and $\text{wdim}(R)$ to denote, respectively, the classical global and weak dimension of $R$.

In [4], the authors introduced a new global dimension over a ring $R$, $r.IFD(R)$, defined as

$$r.IFD(R) = \sup\{\text{fd}_R(I) \mid I \text{ is a right injective } R\text{-module}\}.$$

For such dimension, Ding and Chen gave a various characterizations (see [4]).

For an $R$-module $M$, let $\text{IF-d}_R(M)$ denote the smallest integer $n \geq 0$ such that $\text{Tor}^i_R(I, M) = 0$ for all $i > n$ and all right injective $R$-module $I$ and call $\text{IF-d}_R(M)$ the IF-dimension of $M$. If no such $n$ exists, set $\text{IF-d}_R(M) = \infty$. If $\text{IF-d}_R(M) = 0$, then $M$ will be called IF-module.

In this paper, we investigate the IF-dimension and we give the following new characterization $\text{IFD}(R) = \sup\{\text{IF-d}(R/I) \mid I \text{ is a left ideal of } R\}$.

Recall that a subclass $\mathcal{X}$ of $R$-modules is called projectively resolving if $\mathcal{X}$ contain all projective modules, and for every short exact sequence $0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$ with $X'' \in \mathcal{X}$ the conditions $X' \in \mathcal{X}$ and $X \in \mathcal{X}$ are equivalent (see [9, Definition 1.1]). Also an $R$-module is called FP-injective if, for all finitely presented $N$, $\text{Ext}^1_R(N, M) = 0$. The FP-injective dimension of $M$, denoted by $\text{FP-id}_R(M)$,
is defined to be the least positive integer \( n \) such that \( \text{Ext}_R^{n+1}(N, M) = 0 \) for all finitely presented \( R \)-module \( N \). If no such \( n \) exists, set FP-id\(_R\)(\( M \)) = \( \infty \) (see [4]). The character module \( \text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z}) \) is denoted by \( M^+ \) and \( E(M) \) denote the envelope injective of the \( R \)-module \( M \).

Recall that a short exact sequence of right \( R \)-modules

\[
0 \to A \to B \to C \to 0
\]

is pure exact in the sense of Cohn ([2]) if for all (or equivalently for all finitely presented) left \( R \)-module \( K \) we have the exactness of

\[
0 \to A \otimes_R K \to B \otimes_R B \to C \otimes_R K \to 0.
\]

In this case we say also that \( A \) is a pure submodule of \( B \). For instance, a right \( R \)-module is FP-injective if, and only if, it is absolutely pure (i.e.; it is a pure submodule of every module containing it as a submodule); see [5, 8].

**Lemma 1.1 ([3, Theorem 3.1]).** For any ring \( R \), a sequence of right \( R \)-modules

\[
0 \to A \to B \to C \to 0
\]

is pure exact if, and only if, the sequence of character modules

\[
0 \to C^+ \to B^+ \to A^+ \to 0
\]

is split exact.

2. **Main results**

We start with the following proposition:

**Proposition 2.1.** The class of all IF module is projectively resolving, closed under arbitrary direct sums, and under direct summands.

**Proof.** Clearly, every flat (and then every projective) module is IF-module. Now, let \( 0 \to M \to M' \to M'' \to 0 \) be an exact sequence of \( R \)-modules. If \( M'' \) is an IF-module. It is trivial that \( M' \) is IF-module if, and only if, \( M \) is IF-module by using the functor \( \text{Tor}_i^R(E, -) \) with \( E \) right injective and the long suite exact sequence of homology. Since \( \text{Tor}_i^R(E, -) \) commutes with direct sums the other assertions follows immediately. \( \square \)

We have the following characterization of IF-dimension:

**Proposition 2.2.** For any \( R \)-module \( M \) and any positive integer \( n \), the following are equivalent:

1. FP-d\(_R\)(\( M \)) \( \leq n \).
2. \( \text{Tor}_i^R(E, M) = 0 \) for all \( i > n \) and all right injective \( R \)-module \( E \).
3. \( \text{Tor}_i^R(E, M) = 0 \) for all \( i > n \) and all right FP-injective \( R \)-module \( E \).
4. \( \text{Tor}_i^R(E, M) = 0 \) for all \( i > n \) and all right \( R \)-module \( E \) with finite FP-injective dimension.
5. \( \text{Tor}_i^R(F^+, M) = 0 \) for all \( i > n \) and all \( R \)-module \( F \) with finite flat dimension.
(6) If \( 0 \to F_n \to F_{n-1} \to \ldots \to F_0 \to M \to 0 \) is exact with \( F_0, \ldots, F_{n-1} \) are IF-modules, then \( F_n \) is also an IF-module.

**Proof.** The proof of (1) \( \iff \) (2) \( \iff \) (6) is standard homological algebra fare.

(2) \( \Rightarrow \) (3). Consider an arbitrary right FP-injective \( R \)-module \( K \). From [3, Theorem 3.1], the short pure exact sequence of right \( R \)-modules \( 0 \to K \to E(K) \to E(K)/K \to 0 \) where \( E(K) \) is the injective envelope of \( K \) induce the split exact sequence \( 0 \to (E(K)/K)^+ \to E(K)^+ \to K^+ \to 0 \). Thus, \( K^+ \) is a direct summand of \( E(K)^+ \). On the other hand, by adjointness, \( \text{Ext}^i_K(M,E(K)^+) \cong (\text{Tor}_i^K(E(K),M))^+ = 0 \). Thus, \( (\text{Tor}_i^K(K,M))^+ = \text{Ext}^i_K(M,K^+) = 0 \), and so \( \text{Tor}_i^K(K,M) \) vanishes as desired.

(3) \( \Rightarrow \) (4). The proof will be by induction on \( m = \text{FP-id}_R(K) \). The induction start is clear, from (3). If \( m > 0 \), pick a short exact sequence \( 0 \to K \to E(K) \to E(K)/K \to 0 \) where \( E(K) \) is the envelope injective of \( K \). Then, by a standard homological algebra we can see that \( \text{FP-id}_R(E(K)/K) = m - 1 \). So, we have, for all \( i > n \), the exact sequence

\[
\text{Tor}_i^{i+1}(E(K)/K,M) \to \text{Tor}_i^K(K,M) \to \text{Tor}_i^K(E(K),M).
\]

Clearly, \( \text{Tor}_i^{i+1}(E(K)/K,M) \) and \( \text{Tor}_i^K(E(K),M) \) vanishes by the induction hypothesis conditions and since \( E(K) \) is injective (and so FP-injective), respectively. Thus, \( \text{Tor}_i^K(K,M) = 0 \), as desired.

(4) \( \Rightarrow \) (2). Obvious since every injective module is FP-injective.

(4) \( \Rightarrow \) (5). Set \( m = \text{fd}_K(F) < \infty \). Then, by [6, Theorem 2.1], \( m = \text{id}_K(F^+) = \text{FP-id}_R(F^+) \). Thus, by hypothesis, \( \text{Tor}_i^K(F^+,M) = 0 \) for all \( i > n \).

(5) \( \Rightarrow \) (2). Let \( I \) be a right injective \( R \)-module. There exist a flat \( R \)-module \( F \) such that \( F \to I^+ \to 0 \) is exact. Then, \( 0 \to I^+ \to F^+ \) is exact. But \( 0 \to I \to I^+ \) is exact (by [7, Proposition 3.52]). Then, \( 0 \to I \to F^+ \) is exact and then \( I \) is a direct summand of \( F^+ \). So, \( \text{Tor}_i^K(I,M) = 0 \) as a direct summand of \( \text{Tor}_i^K(F^+,M) \).

Clearly every flat module is IF-module but the inverse implication is not true in the general case as shown by the following example.

**Example 2.3.** Consider the local quasi-Frobenius ring \( R = k[X]/(X^2) \) where \( k \) is a field, and denote by \( \overline{X} \) the the residue class in \( R \) of \( X \). Then, \( \overline{X} \) is an IF \( R \)-module but not flat.

**Proof.** Since \( R \) is quasi-Frobenius, every right injective module \( E \) is projective (and so flat). Then, \( \text{Tor}_i^K(E,\overline{X}) = 0 \) for all \( i > 0 \). Thus, \( \overline{X} \) is an IF \( R \)-module. Now, if we suppose that \( \overline{X} \) is flat. Then, it is projective since it is finitely presented. But \( R \) is also local. So, \( \overline{X} \) must be a free which is absurd by the fact that \( \overline{X}^2 = 0 \). So, we conclude that \( \overline{X} \) is not flat, as desired.
**Proposition 2.4.** Let \( R \) be any ring and \( n \) a positive integer. The following are equivalents:

(1) \( r.\text{IFD}(R) \leq n \).
(2) \( \text{IF-d}_R(M) \leq n \) for every \( R \)-module \( M \).
(3) \( \text{IF-d}_R(M) \leq n \) for every finitely generated \( R \)-module \( M \).
(4) \( \text{IF-d}_R(R/I) \leq n \) for every ideal \( I \).

**Proof.** (1)\(\Rightarrow\)(2). Let \( M \) be an arbitrary \( R \)-module and \( E \) an injective right \( R \)-module. Since \( r.\text{IFD}(R) \leq n \), \( \text{fd}_R(E) \leq n \). Then, \( \text{Tor}^i_R(E,M) = 0 \) for all \( i > n \). Thus, \( \text{IF-d}_R(M) \leq n \), as desired.

(2)\(\Rightarrow\)(3)\(\Rightarrow\)(4). Obvious.

(4)\(\Rightarrow\)(1). Let \( E \) be a right injective \( R \)-module. Since \( \text{IF-d}_R(R/I) \leq n \) for every ideal \( I \), we have \( \text{Tor}_i^R(E,R/I) = 0 \) for all \( i > n \). Thus, from [11, Lemma 9.18], \( \text{fd}_R(E) \leq n \).

**Corollary 2.5** (Theorem 3.5, [4]). For any ring \( R \) and any positive integer \( n \), the following are equivalent:

(1) \( \text{IFD}(R) \leq n \).
(2) \( \text{fd}_R(E) \leq n \) for every FP-injective right module \( E \).
(3) \( \text{fd}_R(E) \) for every right \( R \)-module with finite FP-injective dimension.
(4) \( \text{fd}_R(F^+) \leq n \) for every \( R \)-module with finite flat dimension.

**Proof.** Follows immediately from Propositions 2.2 and 2.4.

For a left coherent ring and a finitely presented modules we get:

**Proposition 2.6.** Let \( R \) be a left coherent ring and \( M \) a finitely presented \( R \)-module. Then, the following are equivalent for any positive integer \( n \).

(1) \( r.\text{IF-d}_R(M) \leq n \)
(2) \( \text{Ext}^i_R(M,R) = 0 \) for all \( i > n \).
(3) \( \text{Tor}^i_R(Q,M) = 0 \) for all \( i > n \) where \( Q \) is any injective cogenerator in the class of the right \( R \)-module.
(4) \( \text{Tor}^i_R(E(S),M) = 0 \) for all \( i > n \) and for all right simple module \( S \).

**Proof.** (1)\(\Rightarrow\)(3). Trivial.

(3)\(\Rightarrow\)(1). Let \( I \) be a right injective \( R \)-module. Since \( Q \) is an injective cogenerator in the class of the right \( R \)-module, \( I \) can be embedded in some product of \( Q \)’s. Thus, \( I \) is a direct summand of \( \prod Q \). Using [10, Lemma 1], \( \text{Tor}^i_R\left(\prod Q, M\right) = \prod \text{Tor}^i_R(Q,M) = 0 \) for all \( i > n \). Thus, \( \text{Tor}^i_R(I,M) = 0 \) for all \( i > n \), as desired.

(1)\(\Rightarrow\)(2). Since \( R \) is left coherent, \( (\text{Ext}^i_R(M,R))^+ = \text{Tor}^i_R(R^+,M) = 0 \) for all \( i > 0 \). Thus, \( \text{Ext}^i_R(M,R) = 0 \).
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(2)$\Rightarrow$(1). Follows from the implication (3)$\Rightarrow$(1) since $R^+$ is an injective cogenerator in the class of right $R$-module and since $(\text{Ext}^1_R(M, R))^+ = \text{Tor}^1_R(R^+, M)$ (recall that $R$ is left coherent).

(1)$\Rightarrow$(4). Trivial.

(4)$\Rightarrow$(1). Using [1, Proposition 18.15], $Q = \prod E(S)$ (direct product of the envelope injective of simple right modules) is an injective cogenerator in the class of right $R$-modules. On the other hand, from [10, Lemma 1], $\text{Tor}_d^1\left(\prod E(S), M\right) = \prod \text{Tor}_d^1(E(S), M)$ = 0. Thus, this implication follows from (3)$\Rightarrow$(1).

\textbf{Proposition 2.7.} For any $R$-module $M$, we have $\text{IF-d}_R(M) \leq \text{fd}_R(M)$ with equality if $\text{fd}_R(M)$ is finite.

\textbf{Proof.} The first inequality follows from the fact that every flat module is an IF-module. Now, set $\text{fd}_R(M) = n < \infty$ and suppose that $\text{IF-d}_R(M) = m < n$. Thus, there is a right $R$-module $N$ such that $\text{Tor}_m^R(N, M) \neq 0$. Clearly, $N$ cannot be injective. Thus, we can consider the exact sequence $0 \to N \to E(N) \to E(N)/N \to 0$ where $E(N)$ is the envelope injective of $N$. Then, we have

$0 = \text{Tor}_m^R(E(N), M) \to \text{Tor}_{m+1}^R(E(N)/N, M) \to \text{Tor}_m^R(N, M) \to \text{Tor}_m^n(E(N), M) = 0$

Therefore, $\text{Tor}_m^R(E(N)/N, M) \neq 0$. Absurd, since $\text{fd}_R(M) = n$.

The proof of the next proposition is standard homological algebra.

\textbf{Proposition 2.8.} Let $0 \to A \to B \to C \to 0$ an exact sequence of $R$-modules. If two of $\text{IF-d}_R(A)$, $\text{IF-d}_R(B)$, and $\text{IF-d}_R(C)$ are finite, so is the third. Moreover,

(1) $\text{IF-d}_R(B) \leq \sup\{\text{IF-d}_R(A), \text{IF-d}_R(C)\}$.

(2) $\text{IF-d}_R(A) \leq \sup\{\text{IF-d}_R(B), \text{IF-d}_R(C) - 1\}$.

(3) $\text{IF-d}_R(C) \leq \sup\{\text{IF-d}_R(B), \text{IF-d}_R(A) + 1\}$.

The next corollary is an immediate consequence of Proposition 2.8.

\textbf{Corollary 2.9.} Let $R$ be a ring, $0 \to A \to B \to C \to 0$ an exact sequence of $R$-modules. If $B$ is an IF-module and $\text{IF-d}(C) > 0$ then, $\text{IF-d}(C) = \text{IF-d}(A) + 1$.

\textbf{Proposition 2.10} (Flat base change). Consider a flat homomorphism of commutative rings $R \to S$ (that is, $S$ is flat as an $R$-module). Then for any $R$-module $M$ we have the inequality, $\text{IF-d}_S(M \otimes_R S) \leq \text{IF-d}_R(M)$.

\textbf{Proof.} Suppose that $\text{IF-d}_R(M) \leq n$. If $I$ is an injective $S$-module, then since $S$ is $R$-flat, $I$ is also an injective $R$-module. On the other hand, from [11, Theorem 11.64], we have $\text{Tor}_i^S(I, M \otimes_R S) \cong \text{Tor}_i^R(I, M) = 0$ for all $i > n$. Thus, $\text{IF-d}_S(M \otimes_R S) \leq n$, as desired.
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