Communications in Mathematics and Applications
Volume 1 (2010), Number 1, pp. 59–70

http://www.rgnpublications.com

Five Mappings in Connection to Hadamard's Inequality

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Abstract. In this paper we point out five new inequalities of the Hadamard's type and use a simple new technique in the proof.

1. Introduction

Let $f : I \subset \mathfrak{R} \to \mathfrak{R}$ be a convex mapping of the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known in the literature as Hadamard's inequality. In [1], Fejer generalized the inequality (1.1) by proving that if $g : [a, b] \to \mathfrak{R}$ is nonnegative, integrable and symmetric to $x = \frac{a+b}{2}$, and if f is convex on $[a, b]$, then

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx &\le \int_a^b f(x)g(x)dx \\ &\le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx. \end{aligned} \tag{1.2}$$

A *positive function* f is said to be r-convex on $[a, b]$ if for all $x, y \in [a, b]$, and $\lambda \in [0, 1]$,

$$f(\lambda x + (1-\lambda)y) \le \begin{cases} (\lambda f^r(x) + (1-\lambda)f^r(y))^{\frac{1}{r}}, & r \ne 0, \\ f^{\lambda}(x)f^{1-\lambda}(y), & r = 0. \end{cases}$$

The 0-convex functions are simply the log-convex functions and 1-convex functions are the ordinary convex functions.

We define the following mappings:

$$m(t) = \frac{1}{b-a}\int_a^b f^r\left(tx + (1-t)\frac{a+b}{2}\right)dx, \quad r \ne 0,$$

Key words and phrases. Hadamard's inequality.

$$n(t) = \frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dx,$$

$$M(t) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f^r(tx+(1-t)y)dxdy, \quad r\neq 0,$$

$$N(t) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)dxdy,$$

$$L(t) = \left(\frac{1}{(b-a)^2}\int_a^b\int_a^b f^r(tx+(1-t)y)dxdy\right)^{\frac{1}{r}}.$$

Recently, Dragomir [3], proved the following results

Theorem 1.1. *Let $f:[a,b]\to\Re$ be convex and let $H:[0,1]\to\Re$ be defined by*

$$H(t)=\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dx.$$

Then,

(i) *H is convex in $[0,1]$.*
(ii) *We have*

$$\inf_{t\in[0,1]} H(t)=H(0)=f\left(\frac{a+b}{2}\right),$$

$$\sup_{t\in[0,1]} H(t)=H(1)=\frac{1}{b-a}\int_a^b f(x)dx.$$

(iii) *H is convex in $[0,1]$.*

Theorem 1.2. *Let $f:[a,b]\to\Re$ be convex and let $F:[0,1]\to\Re$ be defined by*

$$F(t)=\frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)dxdy.$$

Then,

(i) $F\left(\frac{1}{2}+t\right)=F\left(\frac{1}{2}-t\right)$ *for all t in* $\left[0,\frac{1}{2}\right]$.
(ii) *F is convex on $[0,1]$.*
(iii) *We have*

$$\sup_{t\in[0,1]} F(t)=F(0)=F(1)=\frac{1}{b-a}\int_a^b f(x)dx,$$

$$\inf_{t\in[0,1]} F(t)=F\left(\frac{1}{2}\right)=\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy.$$

(iv) *The following inequality is valid*

$$f\left(\frac{a+b}{2}\right)\leq F\left(\frac{1}{2}\right).$$

(v) *F decreases monotonically on $[0,1/2]$ and increases monotonically on $[1/2,1]$.*

(vi) *We have the inequality*

$$H(t) \le F(t) \quad \text{for all} \quad t \in [0,1].$$

2. Main Result

We prove the following

Theorem 2.1. *If f is r-convex $(r \neq 0)$, then $m(t)$ has the following properties:*

(i) *$m(t)$ is convex on $[0,1]$.*

(ii) $$\inf_{t\in[0,1]} m(t) = m(0) = f^r\left(\frac{a+b}{2}\right),$$

$$\sup_{t\in[0,1]} m(t) = m(1) = \frac{1}{b-a}\int_a^b f^r(x)dx.$$

(iii) *m increases on $[0,1]$.*

Proof. **(i)** Let $t_1, t_2 \in [0,1]$, $\alpha, \beta \ge 0$, $\alpha + \beta = 1$. Then

$$\begin{aligned}
&m(\alpha t_1 + \beta t_2)\\
&= \frac{1}{b-a}\int_a^b f^r\left((\alpha t_1 + \beta t_2)x + (1 - \alpha t_1 - \beta t_2)\frac{a+b}{2}\right)dx\\
&= \frac{1}{b-a}\int_a^b f^r\left(\alpha\left(t_1 x + (1-t_1)\frac{a+b}{2}\right) + \beta\left(t_2 x + (1-t_2)\frac{a+b}{2}\right)\right)dx\\
&\le \frac{1}{b-a}\int_a^b \left(\alpha f^r\left(t_1 x + (1-t_1)\frac{a+b}{2}\right) + \beta f^r\left(t_2 x + (1-t_2)\frac{a+b}{2}\right)\right)dx\\
&= \alpha m(t_1) + \beta m(t_2).
\end{aligned}$$

(ii)

$$\begin{aligned}
f^r\left(\frac{a+b}{2}\right) &= \frac{1}{b-a}\int_a^b f^r\left(\frac{\begin{pmatrix} tx + (1+t)\frac{a+b}{2} \\ + t(a+b-x) + (1-t)\frac{a+b}{2}\end{pmatrix}}{2}\right)dx\\
&\le \frac{1}{b-a}\int_a^b \left(\frac{1}{2}\left(f^r\left(tx + (1-t)\frac{a+b}{2}\right)\right)\right.\\
&\qquad \left. + \frac{1}{2}\left(f^r\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right)\right)\right)dx\\
&= \frac{1}{b-a}\left(\frac{1}{2}\int_a^b f^r\left(tx + (1-t)\frac{a+b}{2}\right)dx\right.\\
&\qquad \left. + \frac{1}{2}\int_a^b f^r\left(tx + (1-t)\frac{a+b}{2}\right)dx\right)\\
&= m(t).
\end{aligned}$$

By the r-convexity of f, we have

$$\begin{aligned} m(t) &\leq \frac{1}{b-a}\int_a^b f\left(tf^r(x)+(1-t)f^r\left(\frac{a+b}{2}\right)\right)dx \\ &\leq \frac{1}{b-a}\left(t\int_a^b f^r(x)dx+(1-t)m(1)\right) \\ &\leq \frac{1}{b-a}\left(t\int_a^b f^r(x)dx+(1-t)\int_a^b f^r(x)dx\right) \\ &= \frac{1}{b-a}\int_a^b f^r(x)dx \\ &= m(1). \end{aligned}$$

Therefore, we have

$$\inf_{t\in[0,1]} m(t)=m(0)=f^r\left(\frac{a+b}{2}\right),$$

$$\sup_{t\in[0,1]} m(t)=m(1)=\frac{1}{b-a}\int_a^b f^r(x)dx.$$

(iii) Let $t_1,t_2\in(0,1)$ with $t_2>t_1$. By the convexity of m, we have

$$\frac{m(t_2)-m(t_1)}{t_2-t_1}\geq\frac{m(t_1)-m(0)}{t_1-0}\geq 0,$$

which implies that $m(t_2)\geq m(t_1)$. □

Theorem 2.2. *If f is log-convex, then $n(t)$ has the following properties:*

(i) $\log n(t)$ *is convex on* $[0,1]$.

(ii) $\inf_{t\in[0,1]} \log n(t)=\log n(0)=\log f\left(\frac{a+b}{2}\right)$,

$\sup_{t\in[0,1]} \log n(t)=\log n(1)=\log\left(\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right)$.

(iii) $\log n(t)$ *increases on* $[0,1]$.

Proof. **(i)** In this case we have to show that log n is convex. For this let $t_1,t_2\in[0,1]$, $\alpha,\beta\geq 0$, $\alpha+\beta=1$. Then

$$\begin{aligned} n(\alpha t_1+\beta t_2) &= \frac{1}{b-a}\int_a^b f\left((\alpha t_1+\beta t_2)x+(1-\alpha t_1-\beta t_2)\frac{a+b}{2}\right)dx \\ &= \frac{1}{b-a}\int_a^b f\left(\alpha\left(t_1x+(1-t_1)\frac{a+b}{2}\right)\right. \\ &\qquad\left.+\beta\left(t_2x+(1-t_2)\frac{a+b}{2}\right)\right)dx \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{b-a}\int_a^b \left(f^{\alpha}\left(t_1x+(1-t_1)\frac{a+b}{2}\right)f^{\beta}\left(t_2x+(1-t_2)\frac{a+b}{2}\right)\right)dx\\
&\le \frac{1}{b-a}\left(\int_a^b f\left(t_1x+(1-t_1)\frac{a+b}{2}\right)dx\right)^{\alpha}\\
&\qquad\times\left(\int_a^b f\left(t_2x+(1-t_2)\frac{a+b}{2}\right)dx\right)^{\beta}\\
&= \left(\frac{1}{b-a}\int_a^b f\left(t_1x+(1-t_1)\frac{a+b}{2}\right)dx\right)^{\alpha}\\
&\qquad\times\left(\frac{1}{b-a}\int_a^b f\left(t_2x+(1-t_2)\frac{a+b}{2}\right)dx\right)^{\beta}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
\log n(\alpha t_1+\beta t_2) &\le \alpha\log\left(\frac{1}{b-a}\int_a^b f\left(t_1x+(1-t_1)\frac{a+b}{2}\right)dx\right)\\
&\qquad+\beta\log\left(\frac{1}{b-a}\int_a^b f\left(t_2x+(1-t_2)\frac{a+b}{2}\right)dx\right)\\
&= \alpha\log n(t_1)+\beta\log n(t_2)
\end{aligned}
$$

(ii) We have

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &= \frac{1}{b-a}\int_a^b f\left(\frac{\begin{pmatrix} tx+(1-t)\frac{a+b}{2}\\ +t(a+b-x)+(1-t)\frac{a+b}{2}\end{pmatrix}}{2}\right)dx\\
&\le \frac{1}{b-a}\int_a^b \left(f^{\frac12}\left(tx+(1-t)\frac{a+b}{2}\right)f^{\frac12}\right.\\
&\qquad\left.\times\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)\right)dx\\
&\le \frac{1}{b-a}\left(\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dx\right)^{\frac12}\\
&\qquad\times\left(\int_a^b f\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)dx\right)^{\frac12}\\
&= \frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dx = n(t),
\end{aligned}
$$

which implies

$$
\log f\left(\frac{a+b}{2}\right) \le \log n(t).
$$

Also,

$$\begin{aligned}
n(t) &\le \frac{1}{b-a}\int_a^b f^t(x) f^{1-t}\left(\frac{a+b}{2}\right)dx \\
&\le \frac{1}{b-a}\left(\int_a^b f(x)dx\right)^t\left(\int_a^b f\left(\frac{a+b}{2}\right)dx\right)^{1-t} \\
&= \left(\frac{1}{b-a}\int_a^b f(x)dx\right)^t f^{1-t}\left(\frac{a+b}{2}\right) \\
&\le n^t(1)n^{1-t}(!) \\
&= n(1),
\end{aligned}$$

and this implies

$$\log n(t) \le \log n(1).$$

Therefore, we have

$$\inf_{t\in[0,1]} \log n(t) = \log n(0) = \log f\left(\frac{a+b}{2}\right),$$

$$\sup_{t\in[0,1]} \log n(t) = \log n(1) = \log\left(\frac{1}{b-a}\int_a^b f(x)dx\right).$$

(iii) This follows exactly as in the case of Theorem 2.1. □

Theorem 2.3. *If f is r-convex $(r \neq 0)$, then $M(t)$ has the following properties:*

(i) *$M(t)$ is convex on $[0,1]$.*

(ii) $\inf_{t\in[0,1]} M(t) = M\left(\frac{1}{2}\right) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f^r\left(\frac{x+y}{2}\right)\mathrm{dx}\,\mathrm{dy},$

$\sup_{t\in[0,1]} M(t) = M(1) = M(0) = \frac{1}{b-a}\int_a^b f^r(x)\mathrm{dx}.$

(iii) $M\left(\frac{1}{2}-t\right) = M\left(\frac{1}{2}+t\right),\ t \in \left[0,\frac{1}{2}\right].$

(iv) *M decreases on $\left[0,\frac{1}{2}\right]$ and increases on $\left[\frac{1}{2},1\right]$.*

(v) $f^r\left(\frac{a+b}{2}\right) \ge M\left(\frac{1}{2}\right).$

(vi) $m(t) \le M(t).$

Proof. **(i)** It is similar to that given in Theorem 2.1.

(ii) $$M\left(\frac{1}{2}\right) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f^r\left(\frac{x+y}{2}\right)dx\,dy$$

$$
\begin{aligned}
&= \frac{1}{(b-a)^2}\int_a^b\int_a^b f^r\left(\frac{tx+(1-t)y+(1-t)x+ty}{2}\right)dxdy \\
&\le \frac{1}{(b-a)^2}\int_a^b\int_a^b \left(\frac{1}{2}f^r(tx+(1-t)y)+\frac{1}{2}f^r((1-t)x+ty)\right)dxdy \\
&= \frac{1}{(b-a)^2}\int_a^b\int_a^b f^r(tx+(1-t)y)dxdy = M(t).
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
M(t) &\le \frac{1}{(b-a)^2}\int_a^b\int_a^b (tf^r(x)+(1-t)f^r(y))dxdy \\
&= \frac{1}{b-a}\left(t\int_a^b f^r(x)dx+(1-t)\int_a^b f^r(y)dy\right) \\
&= \frac{1}{(b-a)}\int_a^b f(x)dx = M(1).
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
&\inf_{t\in[0,1]} M(t) = M\left(\frac{1}{2}\right) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f^r\left(\frac{x+y}{2}\right)dxdy, \\
&\sup_{t\in[0,1]} M(t) = M(1) = M(0) = \frac{1}{b-a}\int_a^b f^r(x)dx.
\end{aligned}
$$

(iii)
$$
\begin{aligned}
M\left(\frac{1}{2}-t\right) &= \frac{1}{(b-a)^2}\int_a^b\int_a^b f^r\left(\left(\frac{1}{2}-t\right)x+\left(\frac{1}{2}+t\right)y\right)dxdy \\
&= \frac{1}{(b-a)^2}\int_a^b\int_a^b f^r\left(\left(\frac{1}{2}-t\right)y+\left(\frac{1}{2}+t\right)x\right)dydx \\
&= M\left(\frac{1}{2}+t\right).
\end{aligned}
$$

(iv) Let $t_1, t_2 \in \left[\frac{1}{2}, 1\right]$, $t_2 > t_1$. Then, we have, by the convexity of M

$$
\frac{M(t_2)-M(t_1)}{t_2-t_1} \ge \frac{M(t_1)-M(\frac{1}{2})}{t_1-\frac{1}{2}} \ge 0.
$$

Therefore $M(t_2) \ge M(t_1)$. That $M(t)$ decreases on $\left[0, \frac{1}{2}\right]$ follows from (iii).

(v)
$$
\begin{aligned}
f^r\left(\frac{a+b}{2}\right) &= \frac{1}{(b-a)^2}\int_a^b\int_a^b f^r\left(\frac{1}{2}\left(\frac{x+y}{2}+\frac{a+b-x+a+b-y}{2}\right)\right)dxdy \\
&\le \frac{1}{(b-a)^2}\int_a^b\int_a^b \frac{1}{2}\left(f^r\left(\frac{x+y}{2}\right)+f^r\left(\frac{a+b-x+a+b-y}{2}\right)\right)dxdy
\end{aligned}
$$

$$= \frac{1}{(b-a)^2}\int_a^b\int_a^b f^r\left(\frac{x+y}{2}\right)dx\,dy = M\left(\frac{1}{2}\right).$$

(vi)
$$m(t) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f^r\left(tx+(1-t)\frac{a+b}{2}\right)dx\,dy$$
$$= \frac{1}{(b-a)^2}\int_a^b\int_a^b f^r\left(\frac{tx+(1-t)y+tx+(1-t)(a+b-y)}{2}\right)dx\,dy$$
$$\le \frac{1}{2}\frac{1}{(b-a)^2}\int_a^b\int_a^b f^r(tx+(1-t)y)dx\,dy$$
$$+ \frac{1}{2}\frac{1}{(b-a)^2}\int_a^b\int_a^b f^r(tx+(1-t)(a+b-y))dx\,dy$$
$$= \frac{1}{(b-a)^2}\int_a^a\int_a^b f^r(tx+(1-t)y)dx\,dy = M(t).$$

$\square$

Theorem 2.4. *If f is log-convex, then $N(t)$ has the following properties*

(i) *$N(t)$ is convex on $[0,1]$.*

(ii) $$\inf_{t\in[0,1]} N(t) = N\left(\frac{1}{2}\right) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dx\,dy,$$

$$\sup_{t\in[0,1]} N(t) = N(1) = N(0) = \frac{1}{b-a}\int_a^b f(x)dx.$$

(iii) $N\left(\frac{1}{2}-t\right) = N\left(\frac{1}{2}+t\right)$, $t\in\left[0,\frac{1}{2}\right]$.

(iv) *N decreases on $\left[0,\frac{1}{2}\right]$ and increases on $\left[\frac{1}{2},1\right]$.*

(v) $f^r\left(\frac{a+b}{2}\right) \le N\left(\frac{1}{2}\right)$.

(vi) $n(t) \le N(t)$.

Proof. **(i)** This is similar to that given in Theorem 2.2.

(ii)
$$N\left(\frac{1}{2}\right) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dx\,dy$$
$$= \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{tx+(1-t)y+(1-t)x+ty}{2}\right)dx\,dy$$
$$\le \frac{1}{(b-a)^2}\int_a^b\int_a^b \left(f^{\frac{1}{2}}(tx+(1-t)y)f^{\frac{1}{2}}((1-t)x+ty)\right)dx\,dy$$

$$\leq \frac{1}{(b-a)^2}\left(\int_a^b\int_a^b f(tx+(1-t)y)dx\,dy\right)^{\frac{1}{2}}$$

$$\times\left(\int_a^b\int_a^b f(ty+(1-t)x)\,dydx\right)^{\frac{1}{2}}$$

$$= \frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx(1-t)y)dx\,dy = N(t).$$

Also, we have

$$\begin{aligned} N(t) &\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b f^t(x)f^{1-t}(y)dx\,dy \\ &= \frac{1}{(b-a)^2}\left(\int_a^b\int_a^b f(x)dx\,dy\right)^t\left(\int_a^b\int_a^b f(y)dx\,dy\right)^{1-t} \\ &= \frac{1}{(b-a)^2}\int_a^b\int_a^b f(x)dx\,dy \\ &= \frac{1}{(b-a)}\int_a^b f(x)dx = N(1) = N(0). \end{aligned}$$

Therefore, we get

$$\inf_{t\in[0,1]} N(t) = N\left(\frac{1}{2}\right) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dx\,dy,$$

$$\sup_{t\in[0,1]} N(t) = N(1) = N(0) = \frac{1}{b-a}\int_a^b f(x)dx.$$

(iii) and **(iv)** Similar cases are proved before.

(v)
$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{1}{2}\left(\frac{x+y}{2} + \frac{a+b-x+a+b-y}{2}\right)\right)dx\,dy \\ &\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b f^{\frac{1}{2}}\left(\frac{x+y}{2}\right)f^{\frac{1}{2}}\left(\frac{a+b-x+a+b-y}{2}\right)dx\,dy \\ &\leq \frac{1}{(b-a)^2}\left(\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dx\,dy\right)^{\frac{1}{2}} \\ &\qquad\times\left(\int_a^b\int_a^b f\left(\frac{a+b-x+a+b-y}{2}\right)dx\,dy\right)^{\frac{1}{2}} \\ &= \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dx\,dy = N\left(\frac{1}{2}\right). \end{aligned}$$

$$\textbf{(vi)}\quad n(t) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dx\,dy$$
$$\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{tx+(1-t)y+tx+(1-t)(a+b-y)}{2}\right)dx\,dy$$
$$\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b f^{\frac{1}{2}}(tx+(1-t)y)f^{\frac{1}{2}}(tx+(1-t)(a+b-y))dx\,dy$$
$$\leq \frac{1}{(b-a)^2}\left(\int_a^b\int_a^b f(tx+(1-t)y)dx\,dy\right)^{\frac{1}{2}}$$
$$\times\left(\int_a^b\int_a^b f(tx+(1-t)(a+b-y))dx\,dy\right)^{\frac{1}{2}}$$
$$= \frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)dx\,dy = N(t).$$

Theorem 2.5. *If $f:[a,b]\to R$ is convex, then for $r\geq 1$, $L(t)$ has the following properties:*

(i) $L(t)$ *is convex on* $[0,1]$.

(ii) $$\inf_{t\in[0,1]} L(t) = L\left(\frac{1}{2}\right) = \left(\frac{1}{(b-a)^2}\int_a^b\int_a^b f^r\left(\frac{x+y}{2}\right)\mathrm{dx}\,\mathrm{dy}\right)^{\frac{1}{r}},$$
$$\sup_{t\in[0,1]} L(t) = L(1) = L(0) = \left(\frac{1}{b-a}\int_a^b f^r(x)\mathrm{dx}\right)^{\frac{1}{r}}.$$

(iii) $L\left(\frac{1}{2}-t\right) = L\left(\frac{1}{2}+t\right)$, $t\in\left[1,\frac{1}{2}\right]$.

(iv) L *decreases on* $\left[1,\frac{1}{2}\right]$ *and increases on* $\left[\frac{1}{2},1\right]$.

(v) $f^r\left(\frac{a+b}{2}\right)\leq L\left(\frac{1}{2}\right)$.

(vi) $l(t)\leq L(t)$, *where* $l(t)$ *is defined by*
$$l(t) = \left(\frac{1}{(b-a)}\int_a^b f^r\left(tx+(1-t)\frac{a+b}{2}\right)\mathrm{dx}\right)^{\frac{1}{r}}.$$

Proof. **(i)** Let $t_1,t_2\in[0,1]$, and let $\alpha,\beta\geq 0$, with $\alpha+\beta=1$. Then via Minkowski's inequality we have

$$L(\alpha t_1+\beta t_2) = \left(\frac{1}{(b-a)^2}\int_a^b\int_a^b f^r((\alpha t_1+\beta t_2)x+(1-\alpha t_1-\beta t_2)y)dx\,dy\right)^{\frac{1}{r}}$$
$$= \left(\frac{1}{(b-a)^2}\int_a^b\int_a^b f^r(\alpha(t_1+(1-t_1)y)+\beta(t_2x+(1-t_2)y)dx\,dy\right)^{\frac{1}{r}}$$

$$\le \left(\frac{1}{(b-a)^2}\int_a^b\int_a^b (\alpha f(t_1x+(1-t_1)t)+\beta f(t_2x+(1-t_2)y))^r dxdy\right)^{\frac{1}{r}}$$

$$\le \alpha\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b f^r(t_1x+(1-t_1)y)dxdy\right)^{\frac{1}{r}}$$

$$+\beta\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b f^r(t_2x+(1-t_2)y)dxdy\right)^{\frac{1}{r}}$$

$$= \alpha L(t_1)+\beta L(t_2).$$

(ii)
$$L\left(\frac{1}{2}\right) = \left(\frac{1}{(b-a)^2}\int_a^b\int_a^b f^r\left(\frac{x+y}{2}\right)dxdy\right)^{\frac{1}{r}}$$

$$= \left(\frac{1}{(b-a)^2}\int_a^b\int_a^b f^r\left(\frac{tx+(1-t)y+(1-t)x+ty}{2}\right)dxdy\right)^{\frac{1}{r}}$$

$$\le \frac{1}{2}\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b f^r(tx+(1-t)y)dxdy\right)^{\frac{1}{r}}$$

$$+\frac{1}{2}\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b f^r(tx+(1-t)y)dxdy\right)^{\frac{1}{r}}$$

$$= \left(\frac{1}{(b-a)^2}\int_a^b\int_a^b f^r(tx+(1-t)y)dxdy\right)^{\frac{1}{r}} = L(t).$$

Therefore, we have

$$\inf_{t\in[0,1]} L(t) = L\left(\frac{1}{2}\right).$$

Also, we have, by convexity of L

$$L(t) = \left(\frac{1}{(b-a)^2}\int_a^b\int_a^b f^r(tx+(1-t)y)dxdy\right)^{\frac{1}{r}}$$

$$\le t\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b f^r(x)dxdy\right)^{\frac{1}{r}}$$

$$+(1-t)\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b f^r(y)dxdy\right)^{\frac{1}{r}}$$

$$= \left(\frac{1}{(b-a)^2}\int_a^b\int_a^b f^r(x)dxdy\right)^{\frac{1}{r}}$$

$$= \left(\frac{1}{(b-a)}\int_a^b f^r(x)dx\right)^{\frac{1}{r}}.$$

Therefore,

$$\sup_{t\in[0,1]} L(t) = L(1) = L(0).$$

(iii) and **(iv)** may be achieved as before.

(v)
$$f^r\left(\frac{a+b}{2}\right) = \left(\frac{1}{(b-a)^2}\int_a^b\int_a^b f^r\left(\frac{\frac{x+y}{2}+\frac{a+b-x+a+b-y}{2}}{2}\right)dx\,dy\right)^{\frac{1}{r}}$$
$$\le \frac{1}{2}\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b f^r\left(\frac{x+y}{2}\right)dx\,dy\right)^{\frac{1}{r}}$$
$$+\frac{1}{2}\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b f^r\left(\frac{a+b-x+a+b-y}{2}\right)dx\,dy\right)^{\frac{1}{r}}$$
$$= \left(\frac{1}{(b-a)^2}\int_a^b\int_a^b f^r\left(\frac{x+y}{2}\right)dx\,dy\right)^{\frac{1}{r}} = L\left(\frac{1}{2}\right).$$

(vi)
$$l(t) = \left(\frac{1}{(b-a)^2}\int_a^b\int_a^b f^r\left(tx+(1-t)\frac{a+b}{2}\right)dx\,dy\right)^{\frac{1}{r}}$$
$$= \left(\frac{1}{(b-a)^2}\int_a^b\int_a^b f^r\left(\frac{tx+(1-t)y+tx+(1-t)(a+b-y)}{2}\right)dx\,dy\right)^{\frac{1}{r}}$$
$$\le \frac{1}{2}\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b f^r(tx+(1-t)y)dx\,dy\right)^{\frac{1}{r}}$$
$$+\frac{1}{2}\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b f^r(tx+(1-t)(a+b-y))dx\,dy\right)^{\frac{1}{r}}$$
$$= \left(\frac{1}{(b-a)^2}\int_a^b\int_a^b f^r(tx+(1-t)y)dx\,dy\right)^{\frac{1}{r}} = L(t).$$

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Received August 28, 2009
Accepted November 20, 2009