The Boundedness of Cauchy Integral Operator on a Domain Having Closed Analytic Boundary

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Abstract. In this paper, we prove that the Cauchy integral operators (or Cauchy transforms) define continuous linear operators on the Smirnov classes for some certain domain with closed analytic boundary.

Keywords. Smirnov classes; Cauchy integral; Cauchy transform; boundedness; Continuity

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1. Introduction

As usual, we define the Hardy space $H^2 = H^2(\Delta)$ as the space of all functions $f : z \rightarrow \sum_{n=0}^{\infty} a_n z^n$ for which the norm $(\|f\| = \sum_{n=0}^{\infty} |a_n|^2)^{1/2}$ is finite. Here, $\Delta$ is the open unit disc. For a more general simply-connected domain $\Omega$ in the complex plane $\mathbb{C}$ with at least two boundary points, and a conformal mapping $\varphi$ from $\Omega$ onto $\Delta$ (that is, a Riemann mapping function), a function $g$ analytic in $\Omega$ is said to belong to the Smirnov class $E^2(\Omega)$ if and only if $g = (f \circ \varphi)\varphi'_{1/2}$ for some $f \in H^2(\Delta)$ where $\varphi'_{1/2}$ is an analytic branch of the square root of $\varphi'$. The reader is referred to [4, 5, 8, 10], and references therein for a basic account of the subject. $\partial \Delta$ and $\partial \Omega$ will be used to denote the boundary of open unit disc $\Delta$ and the boundary of $\Omega$, respectively.

Suppose that $\Gamma$ is a simple $\sigma$-rectifiable arc (not necessarily closed). The notation $L^p(\Gamma)$ will denote the $L^p$ space of normalized arc length measure on $\Gamma$. Let $\Omega$ denote the complement of
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The Cauchy Integral of a function \( \tilde{f} \) defined on \( \Gamma \) and integrable relative to arc length is defined as:

\[
C_{\Omega} \tilde{f}(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\tilde{f}(\zeta)}{\zeta - z} d\zeta \quad (z \in \Omega).
\]  

(1.1)

\( C_{\Omega} \tilde{f} \) is analytic at each point of \( \Omega \).

If \( \Gamma \) is not closed, then (1.1) defines a single analytic function. If \( \Gamma \) is closed, then \( \Omega \) has two components, the interior and the exterior of \( \Gamma \). Then in each component of (1.1) defines an analytic function.

Recall that a closed analytic curve is a curve \( \gamma = k(\partial \Delta) \) where \( k \) is analytic and conformal in a neighbourhood \( U \) of \( \partial \Delta \). If \( \gamma \) is simple it is called an analytic Jordan curve.

In this paper, we prove:

**Theorem 1.** Suppose that that \( D \) is a bounded simply connected domain and \( \gamma = \partial D \) is a closed analytic curve (e.g. ellipse). Then the Cauchy Integral

\[
C_{D} \tilde{f}(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{\tilde{f}(\zeta)}{\zeta - z} d\zeta
\]

defines a continuous linear operator mapping \( L^2(\partial D) \) into \( E^2(D) \).

**Remark 1.** The result of Theorem 1 is well known in the literature (see e.g., [3], [6]). However, we give a basic and direct proof.

To prove Theorem 1, we need the following lemma and remark:

**Lemma 1 ([4, p. 170]).** Suppose that \( \Omega \) is a simply connected and bounded domain and the boundary \( \Gamma = \partial \Omega \) of \( \Omega \) is a rectifiable Jordan curve. Then

(i) Each \( f \in E^2(\Omega) \) has a nontangential limit function \( \tilde{f} \in L^2(\partial \Omega) \), and

\[
\|f\|_{E^2(\Omega)}^2 = \|\tilde{f}\|_{L^2(\partial \Omega)}^2 = \frac{1}{2\pi} \int_{\partial \Omega} |\tilde{f}(z)|^2 |dz|.
\]

(ii) Each \( f \in E^2(\Omega) \) has a Cauchy representation

\[
f(z) = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{\tilde{f}(\zeta)}{\zeta - z} \Omega^{\zeta} \quad (z \in \Omega).
\]  

(1.2)

In this case, for equation (1.2), we say that Cauchy Integral Formula is valid.

A special case of the above theorem is the following remark. In fact, we prove Theorem 1 using the following remark.

**Remark 2 ([11, p. 423]).** The Cauchy integral formula

\[
C_{\Delta} \tilde{f}(z) = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{\tilde{f}(\zeta)}{\zeta - z} d\zeta
\]

defines a continuous linear operator \( C_{\Delta} : L^2(\partial \Delta) \to E^2(\Delta) \) with \( \|C_{\Delta}\| = 1 \).
Then, we have kernel 

\[ C \]

The operator 

\[ U \]

value singular integral:

\[ \tilde{f} \]

by 

\[ \tilde{C} \]

maps 

concerning boundedness of Cauchy type integrals (see, e.g., [7], [9] and [13]). For the books 

about them and theirs applications as papers and books. For the books concerning Cauchy 

and complex analysis, and have attracted many mathematicians to investigate them.

In fact, there are other types of Cauchy integrals and there have been extensive literature 

about them and theirs applications as papers and books. For the books concerning Cauchy 
type integrals and related subjects (see, for instance, [1], [2], [12], [14] and [15]). For the books 
concerning boundedness of Cauchy type integrals (see, e.g., [7], [9] and [13]).

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### 2. Proof of Theorem [1]

**Proof.** Suppose that \( \varphi \) is a conformal map of \( D \) onto \( \Delta \). Let \( \psi = \varphi^{-1} : \Delta \to D \). Consider the maps 

\[ C_{\Delta} : L^2(\partial\Delta) \to E^2(\Delta) \]

is given by 

\[ C_{\Delta}\tilde{f} = \frac{1}{2\pi i} \int_{\partial\Delta} \frac{\tilde{f}(\xi)}{\xi - z} d\xi \]

and 

\[ \tilde{U}_\psi : L^2(\partial D) \to L^2(\partial\Delta) \]

is given by 

\[ \tilde{U}_\psi f(z) = f(\psi(z))\psi'(z) \]

and 

\[ U_\psi : E^2(\Delta) \to E^2(D) \]

is given by 

\[ U_\psi f(z) = f(\varphi(z))\varphi'(z) \]

and 

\( U_\psi \) and 

\( \tilde{U}_\psi \) are unitary operators. The situation is illustrated in the Figure 1.

![Figure 1](image.png)

**Figure 1.** The maps \( U_\psi \) and \( \tilde{U}_\psi \)

For \( \tilde{f} \in L^2(\partial\Delta) \), we have 

\[ U_\psi C_D \tilde{U}_\psi \tilde{f} = U_\psi C_D (\tilde{f} \circ \varphi) \varphi'^{1/2} \]

and 

\[ C_D (\tilde{f} \circ \varphi(w))\varphi'(w)^{1/2} = \frac{1}{2\pi i} \int_{\partial D} \frac{\tilde{f}(w)\varphi'(w)^{1/2}}{w - \zeta} d\zeta \]

thus 

\[ U_\psi C_D \tilde{U}_\psi \tilde{f}(z) = \psi'(z)^{1/2} \frac{1}{2\pi i} \int_{\partial D} \frac{\tilde{f}(\varphi(\zeta))\varphi'(\zeta)^{1/2}}{\zeta - \psi(z)} d\zeta \]

\[ = \frac{1}{2\pi i} \int_{\partial\Delta} \frac{\tilde{f}(w)\psi'(w)^{1/2}}{\psi'(z)^{1/2}} \frac{\psi'(z)}{w - \psi(z)} d\zeta \]

\( z \in \Delta \).

Then, we have kernel 

\[ K(z,w) = \frac{1}{w - z} H(z,w) \]
where
\[ H(z,w) = \frac{(w-z)\psi'(w)\psi'(z)^{1/2}}{\psi(w) - \psi(z)}. \]

For any \(1 < r\), denote \(\Delta_r\) and \(\sigma_r\) by \(\Delta_r = \{z : |z| < 1\}\) and \(\sigma_r = \partial \Delta_r = \{z : |z| = r\}\).

Since \(\partial D\) is analytic (closed) curve, \(\psi\) is analytic and conformal in a neighbourhood of \(\overline{\Delta}\).

So without loss of generality, we may assume that \(\psi\) is analytic and conformal on \(\Delta_R\) for some \(R > 1\) Then \(\psi\) is analytic and conformal on and inside \(\sigma_r\)' where \(R > r' > 1\).

Choose \(r, s\) such that \(1 < r < s < R\). We shall show that \(H\) is analytic on \(\Delta_r \times \Delta_r\). Hence, because \(r\) is arbitrary, it will follow that \(H\) is analytic on \(\Delta_R \times \Delta_R\).

Fix \(z \in \Delta_R\). Then \(w - F_z(w) = H(z,w)\) is analytic in \(\Delta_R\) except at \(w = z\). But since residue at \(w = z\) is 0, the singularities of \(H\) for \(w = z\) is removable. Hence \(w \to H(z,w)\) is analytic on \(\Delta_R \supset \Delta_s \supset \Delta_r\). We can thus apply Cauchy's integral formula to it, giving
\[ H(z,w) = \frac{1}{2\pi i} \int_{\sigma_r} H(z,v) \frac{1}{v-w} dv \quad \text{(for } w \in \Delta_r \text{ and fixed } z \in \Delta_s, v \in \sigma_r). \quad (2.1) \]

Hence since \(z \in \Delta_s\) is arbitrary, for all \(z \in \Delta_s\) and \(w \in \Delta_r\), equation (2.1) is valid.

By symmetry, for every \(v \in \Delta_R\) the function \(z \to H(z,v)\) is analytic on \(\Delta_R\). Hence
\[ H(z,v) = \frac{1}{2\pi i} \int_{\sigma_s} H(u,v) \frac{1}{u-z} du \quad (z \in \Delta_s, v \in \Delta_R, u \in \sigma_s). \quad (2.2) \]

\(H\) is separately continuous on \(\Delta_R \times \Delta_R\), so it is (jointly) continuous on \(\Delta_R \times \Delta_R\). Substitute the value of \(H(z,v)\) from (2.2) in the integrand of (2.1). Since the function \(H(u,v)\) is continuous, we obtain
\[ H(z,w) = \left(\frac{1}{2\pi i}\right)^2 \int_{\sigma_r} \int_{\sigma_s} \frac{H(u,v)dudv}{(u-z)(v-w)} \quad (z \in \Delta_s, w \in \Delta_r, v \in \sigma_r, u \in \sigma_s). \]

Since \(r\) is arbitrary, we will show that \(H(z,w)\) is analytic on \(\Delta_R \times \Delta_R\). Now,
\[ \frac{1}{(u-z)(v-w)} = \sum_{m,n=0}^{\infty} \frac{z^m}{u^{m+1}} \frac{w^n}{v^{n+1}} \quad (v \in \sigma_r, u \in \sigma_s) \]
and this series is uniformly convergent for \(z \in \Delta_s, w \in \Delta_r, v \in \sigma_r, u \in \sigma_s\). Hence
\[ H(z,w) = \left(\frac{1}{2\pi i}\right)^2 \int_{\sigma_r} \int_{\sigma_s} \sum_{m,n=0}^{\infty} \frac{z^m}{u^{m+1}} \frac{w^n}{v^{n+1}} H(u,v)dudv \quad (z \in \Delta_s, w \in \Delta_r, v \in \sigma_r, u \in \sigma_s). \quad (2.3) \]

Since \(H\) is bounded, the series \(\sum_{m,n=0}^{\infty} \frac{z^m}{u^{m+1}} \frac{w^n}{v^{n+1}} H(u,v)\) is uniformly convergent for \(z \in \Delta_s, w \in \Delta_r, v \in \sigma_r, u \in \sigma_s\).

Because of uniformly convergence, we can integrate the series (2.3) term-by-term and we obtain
\[ H(z,w) = \sum_{m,n=0}^{\infty} z^m w^n \left(\frac{1}{2\pi i}\right)^2 \int_{\sigma_r} \int_{\sigma_s} \frac{H(u,v)}{u^{m+1}v^{n+1}} dudv \quad (z \in \Delta_s, w \in \Delta_r, v \in \sigma_r, u \in \sigma_s) \]
\[ = \sum_{m,n=0}^{\infty} a_{mn} z^m w^n \quad (2.4) \]
where the coefficients \( a_{mn} \) are given by the integral formula
\[
a_{mn} = \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma} \int_{\gamma} \frac{H(u,v)}{u^{m+1}v^{n+1}} \, du \, dv.
\]

Since \( H \) is bounded, we obtain
\[
|a_{mn}| = \left| \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma} \int_{\gamma} \frac{H(u,v)}{u^{n+1}v^{m+1}} \, du \, dv \right| 
\leq \frac{1}{4\pi^2} (2\pi \rho)(2\pi \rho) \|H\|_{\infty} \frac{1}{s^{m+1}\rho^{n+1}} 
\leq s^2 \|H\|_{\infty} \frac{1}{r^{m+1}\rho^{n+1}}
\]
where \( \|H\|_{\infty} = \sup_{u,v \in \gamma} |H(u,v)| < \infty \) and so
\[
\sum_{m,n=0}^{\infty} |a_{mn}| < \infty
\]
(i.e. the series \( \sum_{m,n=0}^{\infty} a_{mn} z^m w^n \) is absolutely convergent). Hence \( \sum_{m,n=0}^{\infty} a_{mn} z^m w^n \) is absolutely convergent on \( \Delta_r \times \Delta_r \). Thus \( H(z,w) \) is analytic on \( \Delta_r (\subseteq \Delta_s) \times \Delta_r \) and so since \( r \) is arbitrary it is analytic on \( \Delta_s \times \Delta_s \).

Now, the series \( H(z,w) = \sum_{m,n=0}^{\infty} a_{mn} z^m w^n \) is uniformly convergent for \( z \in \Delta_s, w \in \Delta_r \). If we set \( A = U\psi C_D \tilde{U}\psi \psi \), then, we have
\[
Af(z) = U\psi C_D \tilde{U}\psi f(z) \quad (f \in L^2(\partial \Delta), z \in \Delta) \]
\[
= \frac{1}{2\pi i} \int_{\partial \Delta} f(w) \frac{\psi(w)^{1/2} \psi'(z)^{1/2}}{\psi(w) - \psi(z)} \, dw \quad (w \in \partial \Delta) 
\]
\[
= \frac{1}{2\pi i} \int_{\partial \Delta} \frac{1}{w - z} H(z,w) f(w) \, dw 
\]= \frac{1}{2\pi i} \int_{\partial \Delta} \frac{1}{w - z} \sum_{m,n=0}^{\infty} a_{mn} z^m w^n f(w) \, dw. \tag{2.5}
\]
In fact, we will show that the sum and the integral in the equation (2.5) can be permutable.

For \( f \in L^2(\partial \Delta), z \in \Delta \), we have
\[
\sum_{m,n=0}^{\infty} \left| \frac{1}{w - z} a_{mn} z^m w^n f(w) \right| \frac{d\omega}{2\pi} \leq \sum_{m,n=0}^{\infty} \left| a_{mn} \right| z^m |w|^n |f(w)| \left| \frac{d\omega}{2\pi} \right| 
\leq \sum_{m,n=0}^{\infty} \left| a_{mn} \right| z^m \left( \int_{\partial \Delta} |f(w)|^2 |d\omega| \right)^{1/2} \left( \int_{\partial \Delta} \left| \frac{1}{w - z} \right|^2 |d\omega| \right)^{1/2} 
\leq \sum_{m,n=0}^{\infty} \left| a_{mn} \right| z^m \|f\| \left( \frac{1}{(1 - |z|^2)^2} \right)^{1/2} < \infty
\]
and by Tonelli Theorem
\[
\frac{1}{2\pi i} \int_{\partial \Delta} \frac{1}{w - z} \sum_{m,n=0}^{\infty} a_{mn} z^m w^n f(w) \, dw = \sum_{m,n=0}^{\infty} a_{mn} z^m \frac{1}{2\pi i} \int_{\partial \Delta} \frac{1}{w - z} w^n f(w) \, dw \quad (f \in L^2(\partial \Delta), z \in \Delta). \tag{2.6}
\]
Now consider the following series of operators
\[ \sum_{m,n=0}^{\infty} a_{mn} M_m C_{\Delta} N_n, \]
where \( M_m : H^2(\Delta) \rightarrow H^2(\Delta) \) is defined by \( M_m f(z) = z^m f(z) \) so that \( \|M_m\| = \|z^m\|_\infty = 1 \), and \( N_n : L^2(\partial \Delta) \rightarrow L^2(\partial \Delta) \) is defined by \( N_n f(z) = z^n f(z) \), so that \( \|N_n\| = \|z^n\|_\infty = 1 \), then, we have
\[ A f(z) = \sum_{m,n=0}^{\infty} a_{mn} M_m C_{\Delta} N_n f(z) \quad (f \in L^2(\partial \Delta), z \in \Delta). \]
Then the series \( A = \sum_{m,n=0}^{\infty} a_{mn} M_m C_{\Delta} N_n \) is absolutely convergent in operator norm in the space \( B(L^2(\partial \Delta), H^2(\Delta)) \), in fact,
\[ \|M_m C_{\Delta} N_n f\|_{H^2(\Delta)} \leq \|M_m\| \|C_{\Delta}\| \|N_n\| \|f\| \quad (f \in L^2(\partial \Delta)) \]
and
\[ \|M_m C_{\Delta} N_n\| \leq 1 \]
extablishes
\[ \sum_{m,n=0}^{\infty} \|a_{mn} M_m C_{\Delta} N_n\| \leq \sum_{m,n=0}^{\infty} \|a_{mn}\| < \infty \]
(i.e. \( \sum_{m,n=0}^{\infty} a_{mn} M_m C_{\Delta} N_n \) converges absolutely) and
\[ \|A f\|_{E^2(\Delta)} = \left\| \sum_{m,n=0}^{\infty} a_{mn} M_m C_{\Delta} N_n f \right\|_{E^2(\Delta)} \leq \sum_{m,n=0}^{\infty} \|a_{mn} M_m C_{\Delta} N_n f\| \quad (\text{since } \sum_{m,n=0}^{\infty} a_{mn} M_m C_{\Delta} N_n \text{ converges absolutely}) \]
\[ = \sum_{m,n=0}^{\infty} |a_{mn}| \|M_m\| \|C_{\Delta}\| \|N_n\| \|f\| \leq \sum_{m,n=0}^{\infty} |a_{mn}| \|f\| \leq \|f\| \sum_{m,n=0}^{\infty} |a_{mn}| < \infty \]
and so
\[ \|A\| = \left\| \sum_{m,n=0}^{\infty} a_{mn} M_m C_{\Delta} N_n \right\| \leq \sum_{m,n=0}^{\infty} |a_{mn}|. \]
This shows that \( A = U_\psi C_{\tilde{\psi}} \tilde{U}_\psi \) is a continuous operator. It follows that \( C_{\tilde{\psi}} \) is a continuous operator.

**Second Proof of the Continuity of \( A \)**
Since in a Banach space \( X \) (here \( X = B(L^2(\partial \Delta), H^2(\Delta)) \)), every absolutely convergent series is convergent, in the norm of \( X \), to an element of \( X \), \( \sum_{m,n=0}^{\infty} a_{mn} M_m C_{\Delta} N_n \) converges to an element.
\( B \in B(L^2(\partial\Delta), H^2(\Delta)) \), in the sense that
\[
\lim_{m,n \to \infty} \sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl} M_k C_\Delta N_l = B
\]
i.e.
\[
\left\| B - \sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl} M_k C_\Delta N_l \right\| \to 0 \quad \text{as} \quad m, n \to \infty.
\]
Our aim is to show that \( B = A \), (i.e. \( Bf(z) = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{1}{w-z} H(z, w)f(w)dw \)). Fix \( z \in \Delta \) and \( f \in L^2(\partial \Delta) \).
Then
\[
\left\| Bf - \sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl} M_k C_\Delta N_l f \right\| \to 0
\]
i.e.
\[
Bf = \lim_{m,n \to \infty} \sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl} M_k C_\Delta N_l f.
\]
Hence
\[
\sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl} M_k C_\Delta N_l f(z) \to Bf(z)
\]
Now
\[
Bf(z) = \lim_{m,n \to \infty} \sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl} M_k C_\Delta N_l f(z)
\]
\[
= \lim_{m,n \to \infty} \frac{1}{2\pi i} \sum_{k=0}^{m} \sum_{l=0}^{n} \int_{\partial \Delta} \frac{a_{kl} z^k w^l f(w)}{w-z} dw
\]
\[
= \lim_{m,n \to \infty} \frac{1}{2\pi i} \int_{\partial \Delta} \sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl} z^k w^l f(w) dw
\]
\[
= \frac{1}{2\pi i} \int_{\partial \Delta} \sum_{k=0}^{m} \sum_{l=0}^{n} a_{mn} z^m w^l dw \quad \text{(from equation (2.6))}
\]
\[
= \frac{1}{2\pi i} \int_{\partial \Delta} f(w) \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{mn} z^m w^l dw
\]
\[
= \frac{1}{2\pi i} \int_{\partial \Delta} f(w) H(z, w) dw
\]
\[
= Af(z).
\]
So for \( z \in \Delta \) and \( f \in L^2(\partial \Delta) \), \( Bf(z) = Af(z) \). Hence \( B = A \). Therefore, since \( B \) is a continuous operator it follows that \( A = U_\psi C_D \tilde{U}_\varphi \) is a continuous operator. Therefore, \( C_D \) is a continuous operator.

### 3. Conclusion and Further Work

In this paper we proved Theorem 1 in detail by showing the continuity of the Cauchy integral operator \( C_D \). The result of Theorem 1 is well known in the literature, however, we give a basic and direct proof.

By using the methods of the proof of Theorem 1 further work includes investigating more complex scenarios such as choosing the domain \( D \) as (bounded or not) simply connected and \( \partial D \), the boundary of \( D \), as closed analytic \( \sigma \)-rectifiable Jordan curve. Note that an arc or closed curve
γ is called σ-rectifiable if and only if it is a countable union of rectifiable arcs in \( \mathbb{C} \), together with \( (\infty) \) in the case when \( \infty \in \gamma \). For instance, a parabola without \( \infty \) is σ-rectifiable arc, and a parabola with \( \infty \) is σ-rectifiable Jordan curve.

**Competing Interests**

The author declares that he has no competing interests.

**Authors’ Contributions**

The author wrote, read and approved the final manuscript.

**References**


